UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2019 9:00 am to 12:00 pm

PAPER 118

COMPLEX MANIFOLDS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define a *complex manifold* and a *holomorphic vector bundle.*

(b) Let J be the induced almost complex structure on a complex manifold X, considered as a bundle endomorphism $J : (TX)_{\mathbb{C}} \to (TX)_{\mathbb{C}}$. Let $TX^{(1,0)}$ denote the subbundle of $(TX)_{\mathbb{C}}$ whose fibres are eigenspaces of J with eigenvalue i. Show that $TX^{(1,0)}$ naturally admits the structure of a holomorphic vector bundle.

(c) Suppose Y is a smooth analytic hypersurface of X. The *normal bundle* of Y in X is the holomorphic vector bundle $N_{Y/X}$ on Y which is the cokernel of the inclusion $TY^{(1,0)} \hookrightarrow TX^{(1,0)}|_Y$. Prove that

$$N_{Y/X} \cong \mathcal{O}(Y)|_Y.$$

(d) Under the same assumptions as part (c), show that

$$K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y,$$

where K_X and K_Y are the top exterior powers of $T^*X^{(1,0)}$ and $T^*Y^{(1,0)}$ respectively.

2

(a) Let X be a topological space, $\mathcal{F}$ a sheaf on X and $\mathcal{U}$ an open cover of X. Define the groups $C^p(\mathcal{U},\mathcal{F})$, the *boundary maps* $\delta : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ and the *Čech cohomology groups* $\check{H}^j(\mathcal{U},\mathcal{F})$.

(b) Let S be a Riemann surface. A *principal part* at $x \in S$ is a Laurent series, valid in a neighbourhood of x, of the form

$$P = \sum_{k=1}^{n} a_k z^{-k}$$

with $a_k \in \mathbb{C}$ and z a local co-ordinate at x. Let $x_1, \dots, x_d \in S$ and let $P_1, \dots, P_d$ be principal parts at $x_1, \dots, x_d$ respectively.

(i) Show how the $P_1, \dots, P_d$ can be used to form an element $(P_{jl}) \in C^1(\mathcal{U},\mathcal{O})$ for an open cover $\mathcal{U}$ of S such that $\delta(P_{jl}) = 0$.

(ii) Suppose that the Čech cohomology group $H^1(S,\mathcal{O})$ vanishes. Without appealing to Dolbeault's Theorem, show that there is a meromorphic function F on S with principal part P_m at x_m for all $m = 1, \dots, d$.

(c) Let X be a complex manifold. Show that

$$H^q(X,\Omega^p) \cong H^{p,q}_{\bar{\partial}}(X).$$

3

(a) Let $\mathrm{Div}(X)$ be the group of divisors on a complex manifold X. Show that there is an isomorphism

$$\mathrm{Div}(X) \cong H^0(X, \mathcal{K}^*/\mathcal{O}^*).$$

(b) Using the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0,$$

show that there is a morphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ whose kernel is the group of principal divisors on X. *[You may use the result that* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.*]*

(c) Define what it means for a line bundle $L \in \mathrm{Pic}(X)$ to be *ample*, and what it means for L to be *positive*. State the *Kodaira Embedding Theorem*.

(d) Using the Kodaira Embedding Theorem or otherwise, show that if X is projective, any element of $L \in \mathrm{Pic}(X)$ can be written $L \cong H_1 \otimes H_2^*$ with $H_1, H_2 \in \mathrm{Pic}(X)$ very ample line bundles.

(e) When X is projective, show that the morphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ constructed in part (b) is surjective. *[Results from the course may be used provided they are stated correctly.]*

4

Suppose that E is a holomorphic vector bundle with a Hermitian metric h.

(a) Define a *connection* on E. Define what it means for a connection to be *compatible with the Hermitian metric* h, and what it means for a connection to be *compatible with the holomorphic structure.*

(b) In a unitary frame, what conditions on the connection matrix does compatibility with a Hermitian metric impose? In a holomorphic frame, what conditions does compatibility with the holomorphic structure impose? *[It is not necessary to prove your assertions.]*

(c) Prove that there is a unique connection which is compatible with both the Hermitian metric and the holomorphic structure. This connection is called the *Chern connection.*

(d) Let D_1, D_2 be connections on E with $D_1 - D_2 = a$ for some $a \in \mathcal{A}^1_{\mathbb{C}}(\operatorname{End} E)$. Show that the curvatures satisfy

$$F_{D_1} = F_{D_2} + D_2(a) + a \wedge a,$$

where $D_2(a) \in \mathcal{A}^2_{\mathbb{C}}(\operatorname{End} E)$ is defined using the induced connection on $\operatorname{End} E$ and $a \wedge a$ is given by exterior product in the form part and evaluation in $\operatorname{End} E$. *[You may use that for the induced connection on* $\operatorname{End} E$ *one has*

$$(Da)s = D(as) + aDs,$$

for any section s *of* E *and* $a \in \mathcal{A}^1_{\mathbb{C}}(\operatorname{End} E)$.*]*

(e) Suppose that D_1 and D_2 are the Chern connections for Hermitian metrics h_1 and h_2 on E respectively, so that $F_{D_1}, F_{D_2} \in \mathcal{A}^{1,1}_{\mathbb{C}}(\operatorname{End} E)$ and $a \in \mathcal{A}^{1,0}_{\mathbb{C}}(\operatorname{End} E)$. Show that

$$F_{D_1} - F_{D_2} = \bar{\partial} a.$$

[You may assume that the connection on $\operatorname{End} E$ *induced from the Chern connection on* E *is the Chern connection on* $\operatorname{End} E$*]*

5

Let (X, ω) be a compact Kähler manifold.

(a) Define the *Laplacians* $\Delta_d, \Delta_{\bar{\partial}}$ and Δ_{∂}.

(b) Show that $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$. *[You may assume any of the Kähler Identities provided they are stated correctly.]*

A form α is called *harmonic* if $\Delta_d \alpha = 0$. By part (*b*) this is equivalent to $\Delta_{\bar{\partial}} \alpha = 0$. Let $\mathcal{H}^{p,q}(X)$ denote the space of harmonic (p, q)-forms on X with respect to ω.

(c) State the *Hodge Decomposition Theorem* for Kähler manifolds.

(d) Let α be a d-closed (p, q)-form on X. Prove that $\alpha = \partial\bar{\partial}\beta$ for some $(p-1, q-1)$-form β if and only if α is orthogonal to $\mathcal{H}^{p,q}(X)$.

END OF PAPER