UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 9:00 am to 12:00 pm

PAPER 115

DIFFERENTIAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the defining properties of the *exterior derivative* d and show that these properties uniquely determine d. Give an expression of d in local coordinates.

What is an *exact* differential form? By considering the antipodal map on the sphere S^{2n}, prove that every differential form of degree $2n$ on the real projective space $\mathbb{R}P^{2n}$ is exact.

Define the *de Rham cohomology* of a manifold. Prove carefully that the de Rham cohomology $H^k_{dR}(M \times S^1)$ is isomorphic to $H^k_{dR}(M) \oplus H^{k-1}_{dR}(M)$, for every manifold M and every integer $k > 0$.

[*You may assume that for each $m > 0$ a m-form ε on S^m is exact precisely when $\int_{S^m} \varepsilon = 0$.*]

2 Define the terms *immersed submanifold* and *embedded submanifold*. Give an example (with a brief justification) of a submanifold which is immersed but not embedded. Show that if Y is a compact manifold and $\psi : Y \to M$ is an (injective) immersion, then $\psi(Y)$ is an embedded submanifold of M.

Show that if $X \subset M$ is an embedded submanifold of a manifold M and $p \in X$, then on M there is a choice of coordinate neighbourhood U of p with coordinates x_i such that $U \cap X$ is given by the equations $x_j = 0$ for $j = 1, \ldots, k = \dim M - \dim X$.

Is it true that every embedded submanifold $X \subset M$ arises as a pre-image of a regular value of a map from M to some Euclidean space? Justify your answer.

Prove that the n-dimensional torus T^n admits an embedding in $\mathbb{R}^{n+1}$.

3 What is a *Lie group*? Explain briefly what is meant by the logarithmic charts making a subgroup G of $GL(m,\mathbb{C})$ into a Lie group. What is a *Lie algebra*? Suppose that a group of matrices G, as above, is made into a Lie group using logarithmic charts, so the tangent space $\mathfrak{g} = T_I G$ at the identity element is identified with a linear subspace of matrices. Prove that then $\mathfrak{g}$ is a Lie algebra with $[B_1, B_2] = B_1 B_2 - B_2 B_1$ for all $B_1, B_2 \in \mathfrak{g}$.

Define what is meant by a *principal G-bundle* $P \to M$ over a manifold M, where G is a Lie group.

Let the action of $U(n)$ on $GL(n,\mathbb{C})$ be given by the right translations $g \mapsto R_h(g) = gh$, for all $g \in GL(n,\mathbb{C})$ and $h \in U(n)$. Let $R(g) = \{gh : h \in U(n)\}$ denote the orbit of g in this action. By considering a map $g \mapsto \frac{1}{2}\log(g\,g^*)$, or otherwise, show that a family of orbits $V = \cup_{h\in N} R(h)$ is an open subset of $GL(n,\mathbb{C})$ diffeomorphic to $W \times U(n)$. Here $N \subset GL(n,\mathbb{C})$ is some neighbourhood of the identity matrix, W is some neighbourhood of the zero matrix in $H(n) = \{B \in \mathrm{Matr}(n,\mathbb{C}) : B^* = B\}$, $\mathrm{Matr}(n,\mathbb{C})$ is the space of all $n \times n$ complex matrices and B^* denotes the conjugate transpose of B.

Show further that the set of orbits $M = \{R(h) : h \in GL(n,\mathbb{C}\}$ admits a smooth structure such that $\pi : h \in GL(n,\mathbb{C}\} \to R(h) \in M$ is a principal $U(n)$-bundle.

[*Standard properties of the exponential map and the logarithm of matrices may be assumed if accurately stated. You may also assume that* exp *maps a neighbourhood of zero in* $\mathfrak{u}(n) = \{B \in \mathrm{Matr}(n,\mathbb{C}) : B^* = -B\}$ *diffeomorphically onto a neighbourhood of the identity in* $U(n)$ *and that the map* $(B_+, B_-) \mapsto \exp(B_+)\exp(B_-)$ *defines a diffeomorphism from a neighbourhood of zero in* $H(n) \oplus \mathfrak{u}(n)$ *onto a neighbourhood of the identity in* $GL(n,\mathbb{C})$.]

4 (a) What is the *Levi–Civita connection* on a Riemannian manifold? Prove that every Riemannian manifold has a unique Levi–Civita connection.

Now let E be a real vector bundle of rank m over a manifold M. Suppose E is endowed with an inner product on the fibres (varying smoothly with the fibre) and a connection A on E satisfies

$$d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle s_1, d_A s_2\rangle$$

for all sections s_1, s_2 of E. Let $\Phi = \pi^{-1}(U) \to U \times \mathbb{R}^m$ be a local trivialization over an open $U \subset M$ such that the inner product on the fibres is given by the standard Euclidean product on $\mathbb{R}^m$. Show that in this local trivialization the connection matrix of 1-forms $A^\Phi = (A^i_j)$ is skew-symmetric, $A^i_j = -A^j_i$.

(b) Suppose that M is an oriented Riemannian manifold. Define the *Hodge $*$ operator*. If M has an even dimension $2n$, is it true that the linear map defined by $*$ on $\Lambda^n T^*_x M$ at $x \in M$ is always self-adjoint? Give a proof or a counterexample as appropriate.

Define the *Laplace–Beltrami operator* Δ for the differential forms on M. Assuming M is compact, prove that if $\lambda \in \mathbb{R}$ is an eigenvalue of Δ, then $\lambda \geqslant 0$.

[*Standard properties of the volume form of a Riemannian metric may be assumed if accurately stated.*]

END OF PAPER