UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 1:30 pm to 4:30 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, standard results on (co)homology of cell complexes and of manifolds may be used without proof if clearly stated.

1. Let $n > 0$ and $f : S^n \times S^n \to S^n \times S^n$ be a homeomorphism. When n is even, show there are only finitely many possibilities for the action of f on $H^n(S^n \times S^n; \mathbb{Z})$. Is the hypothesis that n be even necessary? Justify your answers.

2. Let $\phi : S^2 \to S^2$ be a map of degree p. Let $X = \mathbb{CP}^2 \cup_\phi e^3$ be obtained by attaching a 3-cell to a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ along ϕ. Let $Y = (S^2 \cup_\phi e^3) \vee S^4$. Prove that $H^*(X; \mathbb{Z}) \cong H^*(Y; \mathbb{Z})$ as rings, but that X and Y are not homotopy equivalent.

3. Let K be the Klein bottle and $Y = \mathbb{RP}^2 \vee S^1$. Prove that $H^*(K; \mathbb{Z}) \cong H^*(Y; \mathbb{Z})$ as rings. Is there a map $Y \to K$ which induces an isomorphism on cohomology? Justify your answer.

2 Let $(A, \leqslant)$ be a partially ordered set with the property that, for any $a, b \in A$, there is some $c \in A$ with $a \leqslant c$ and $b \leqslant c$. Define a *direct system* of abelian groups $\{G_a \mid a \in A\}$ associated to the poset A, and the *direct limit* $\varinjlim_a G_a$ of such a system.

1. Show that, for a sequence $\{a_0, a_1, a_2, \ldots\}$ of integers a_i, the direct limit of the system

$$\mathbb{Z} \xrightarrow{a_0} \mathbb{Z} \xrightarrow{a_1} \mathbb{Z} \xrightarrow{a_2} \mathbb{Z} \xrightarrow{a_3} \cdots$$

   is the subgroup of the rationals $\mathbb{Q}$ consisting of those elements whose denominator divides into some product of the a_j's.

2. Let $X = \cup_{a \in A} X_a$ be a topological space with the property that for every compact set $K \subset X$ there is some $a \in A$ (depending on K) such that K lies inside the subspace X_a. Prove that $H_i(X; \mathbb{Z}) = \varinjlim_a H_i(X_a; \mathbb{Z})$. Deduce that $H_i(X; \mathbb{Z})$ is countable for any i and any open subset $X \subset \mathbb{R}^N$. Construct a connected open subset $X \subset \mathbb{R}^N$ for which $H^1(X; \mathbb{Z})$ is uncountable, justifying your answer.

3. The "mapping telescope" of a sequence of spaces and maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

   is the quotient of the disjoint union $\sqcup_i (X_i \times [i, i+1])$ by the equivalence relation

$$X_i \times [i, i+1] \ni (x_i, i+1) \sim (f_i(x_i), i+1) \in X_{i+1} \times [i+1, i+2].$$

   By considering a suitable mapping telescope, prove that there is a topological space X with reduced homology

$$\widetilde{H}_i(X; \mathbb{Z}) = \begin{cases} \mathbb{Q}_{sq} & i = n \\ 0 & i \neq n \end{cases}$$

   where $\mathbb{Q}_{sq} \subset \mathbb{Q}$ is the subgroup of rationals with square-free denominator.

3 The lens space $L(p) = S^3/(\mathbb{Z}/p)$ is the quotient of the unit sphere $S^3 \subset \mathbb{C}^2$ by the diagonal action of the group of p-th roots of unity. For a coefficient ring R, explain what it means for a real vector bundle $E \to X$ to be *R-oriented*, define the *Thom class*, and state the *Gysin sequence* for such an R-oriented bundle. Hence, or otherwise, compute the cohomology of $L(p)$ (additively) with both $\mathbb{Z}$ and $\mathbb{Z}/p$ coefficients.

For any space X, let $\beta : H^j(X;\mathbb{Z}/p) \to H^{j+1}(X;\mathbb{Z}/p)$ be the composite of the boundary map in the exact sequence associated to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ with the reduction mod p map. Prove that $\beta : H^1(L(p);\mathbb{Z}/p) \to H^2(L(p);\mathbb{Z}/p)$ is an isomorphism.

Let $[L(p)] \in H_3(L(p);\mathbb{Z}/p)$ be a generator coming from a choice of orientation of $L(p)$. Let $a \in H^1(L(p);\mathbb{Z}/p)$ be a generator, and define $t(a) = \langle a \cdot \beta(a), [L(p)] \rangle \in \mathbb{Z}/p$ (where $\cdot$ denotes cup product). If $a' \in H^1(L(p);\mathbb{Z}/p)$ is another generator, how are $t(a')$ and $t(a)$ related? Deduce that if $L(p)$ admits an orientation-reversing homotopy equivalence, then -1 is a quadratic residue mod p, i.e. $-1 \equiv n^2$ modulo p, for some integer n.

4 For a space X with finite-dimensional rational cohomology, and a map $f : X \to X$, we define the *Lefschetz number* of f by

$$L(f) = \sum_i (-1)^i \mathrm{tr}(f^* : H^i(X;\mathbb{Q}) \to H^i(X;\mathbb{Q})).$$

For a closed smooth oriented manifold M, give a formula for the cohomology class ε_Δ, where $\Delta \subset M \times M$ is the diagonal submanifold, and deduce the Lefschetz fixed point theorem. (Standard properties of the classes ε_Y may be assumed if carefully stated.)

Let $M = S^1 \sqcup S^1 \sqcup S^1$ be a disjoint union of three circles, and let $f : M \to M$ be a homeomorphism without fixed points. Prove that the trace of the action of f on $H^1(M;\mathbb{Z})$ belongs to $\{0, 1, 3\}$.

Let $(N, \partial N)$ be a compact smooth manifold with non-empty boundary $\partial N \neq \emptyset$. Let $M = N \cup_{\partial N} N$ be the closed smooth manifold obtained by doubling N. Given a map $f : N \to N$ for which $f(\partial N) \subset \partial N$, state and prove a relationship between the Lefschetz numbers $L(f), L(f|_{\partial N})$ and $L(F)$, where $F : M \to M$ is an obvious "double" of f.

Now let N be the complement of three disjoint open discs in the 2-sphere S^2 and let $f : N \to N$ be a homeomorphism. Prove that if f is fixed-point free, then f cyclically permutes the three boundary components of N.

END OF PAPER