UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2019 9:00 am to 12:00 pm

PAPER 113

ALGEBRAIC GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the germ f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$.

(a) If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact (i.e., every open cover of X has a finite subcover). Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) = A_f$, where A_f as usual denotes the localization of the ring A at the multiplicative subset $\{1, f, f^2, \ldots\}$.

2

(a) Let p be a prime number, $\mathbb{F}_p$ the field with p elements, and $i : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ the canonical morphism. We say a ring A *is of characteristic* p if $p \cdot 1 = 0$ in A. Prove that for a scheme X the following are equivalent:

(i) For every open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ has characteristic p.

(ii) The ring $\Gamma(X, \mathcal{O}_X)$ has characteristic p.

(iii) The scheme morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors through i.

In any of these cases, we say that X is of characteristic p.

(b) Let X be a scheme of characteristic p. Show that there exists a unique morphism $F : X \to X$ which is the identity on scheme-theoretic points and on an open set $U \subseteq X$, $F_U^{\#} : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is given by $F_U^{\#}(a) = a^p$.

(c) Give an example of a scheme X of characteristic p such that morphism F of (b) induces an isomorphism on $\Gamma(X, \mathcal{O}_X)$ but such that F is not an isomorphism. Be sure to explain how to calculate $\Gamma(X, \mathcal{O}_X)$.

In this example, choose a closed point $x \in X$, and calculate the fibre of the morphism F over x, i.e., $X \times_X \operatorname{Spec} \kappa(x)$, where $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

3

(a) Let X be a scheme. Give conditions on X for which the *class group* $\mathrm{Cl}(X)$ is defined, and give the definition of $\mathrm{Cl}(X)$.

(b) Let k be a field of characteristic 0, and let $X = \operatorname{Spec} k[x,y,z]/(xy-z^3)$. Calculate $\mathrm{Cl}(X)$. You may take as given that X is normal.

(c) Let A be a Noetherian integral domain, $x \in A$ an element such that the ideal (x) is prime, and let A_x denote as usual the localization of the ring A at the multiplicative subset $\{1, x, x^2, \ldots\}$. Show that if A_x is a UFD, so is A.

4

(a) Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, $\mathcal{U} := \{U_i \mid i \in I\}$ an open cover of X. Define the i^{th} Čech cohomology group $\check{H}^i(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ with respect to the cover $\mathcal{U}$.

(b) Let k be a field, and let $X = \mathbb{P}^r_k$. Describe, without proof, $H^p(X, \mathcal{O}_X(n))$ for all p and for all n.

Suppose there is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(r+1)} \to \mathcal{T} \to 0$$

for a sheaf of $\mathcal{O}_X$-modules $\mathcal{T}$. Calculate $H^i(X, \mathcal{T})$ for all i.

(c) Let k be a field, let $X = \mathbb{P}^r_k$, and let $Y \subseteq X$ be a hypersurface defined by $Y = V(f)$, for f a homogeneous polynomial of degree d. Suppose $\dim Y \geqslant 1$. Then:

(i) Write down a short exact sequence of sheaves relating $\mathcal{O}_X$, $\mathcal{O}_Y$ and $\mathcal{I}_{Y/X}$, the ideal sheaf of Y in X. Relate $\mathcal{I}_{Y/X}$ to $\mathcal{O}_X(1)$.

(ii) Show for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

(iii) Show Y is connected.

(iv) Show $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < \dim Y$ and all $n \in \mathbb{Z}$.

END OF PAPER