UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2019 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Local LYM inequality. State the LYM inequality, and give two proofs: one using Local LYM and one using maximal chains. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

(i) Let $\mathcal{A} \subset \mathcal{P}([n])$ be a family of sets such that, for each $A \in \mathcal{A}$, there do not exist sets $B, C \in \mathcal{A}$ with $B, C \neq A$ such that $A \subset B$ and $A \supset C$. How large can $\mathcal{A}$ be?

(ii) Suppose that, in the proof of the Erdős-Ko-Rado Theorem using the Kruskal-Katona Theorem, we used Local LYM in place of Kruskal-Katona. What bound would we obtain (on the maximum size of an intersecting family of r-sets from an n-set)?

2

State the Kruskal-Katona Theorem, and give a proof using UV-compressions.

For which pairs (U, V), with U, V disjoint subsets of $[n]$ of the same size, is it the case that for every $1 \leqslant r \leqslant n$, and every family $\mathcal{A} \subset X^{(r)}$, we have $|\partial C_{U,V}(\mathcal{A})| \leqslant |\partial \mathcal{A}|$? Justify your answer.

For each pair (U, V) below, is it the case that for every $1 \leqslant r \leqslant n$, and every left-compressed family $\mathcal{A} \subset X^{(r)}$, we have $|\partial C_{U,V}(\mathcal{A})| \leqslant |\partial \mathcal{A}|$? Justify your answers.

(i) $(U, V) = (345, 126)$

(ii) $(U, V) = (145, 236)$

3

State and prove the vertex-isoperimetric inequality in the grid $[k]^n$.

[You may assume that the theorem you are proving holds in the two-dimensional grid $[k]^2$*.]*

Suppose that every subset of size t in the grid $[4]^n$ has vertex-boundary of size at least s (for some values of t and s). Prove that every subset of size t in the discrete cube Q_{2n} of dimension $2n$ has vertex-boundary of size at least s.

4

State and prove the Uniform Covers Theorem.

State and prove the Bollobás-Thomason Box Theorem.

We say that a non-empty body is *proper* if it is connected and is a finite union of boxes, each of positive volume. Show that if S is a proper body in $\mathbb{R}^3$ such that $|S|^2 = |S_{12}||S_{13}||S_{23}|$ then S is a box.

END OF PAPER