UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2019 1:30 pm to 4:30 pm

PAPER 108

TOPICS IN ERGODIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{B}$ be a set with $\mu(A) > 0$. Prove that there is $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.

State and prove the *Poincaré recurrence theorem.*

In the following question you may use without proof the following result of Furstenberg. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{B}$ be a set with $\mu(A) > 0$. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 0$. Then there is $n > 0$ such that $\mu(A \cap T^{-|P(n)|}A) > 0$.

Let $S \subset \mathbb{Z}$ be a set of positive upper Banach density and let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 0$. Prove that there are $a, b \in S$ and $n \in \mathbb{Z}_{>0}$ such that $b - a = P(n)$.

[*If your argument involves the construction of a measure preserving system, you may use without proof any result of the course to prove the measure preserving property.*]

2

Define *convergence in density* and *Cesàro convergence.*

Let $\{a_n\}$ be a bounded sequence of real numbers and $a \in \mathbb{R}$. Prove that D-lim $a_n = a$ if and only if C-lim $|a_n - a| = 0$.

Let $(X, \mathcal{B}, \mu, T)$ be a weak mixing system. What can you say about the ergodicity of its product with another measure preserving system? Give a condition in terms of $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$, which implies that $(X, \mathcal{B}, \mu, T)$ is weak mixing.

[*State a theorem without proof.*]

Let $(X, \mathcal{B}, \mu, T)$ be a weak mixing system. Prove that the Koopman operator U_T has no non-constant eigenfunctions.

Prove that a measure preserving system $(X, \mathcal{B}, \mu, T)$ is weak mixing if and only if for all sets $A, B, C \in \mathcal{B}$ of positive measure, there is $n \geqslant 1$ such that $\mu(T^{-n}A \cap B) \neq 0$ and $\mu(T^{-n}A \cap C) \neq 0$.

[*In this last part of the question, you may use without proof any characterization of weak mixing from the course.*]

3

Define the *entropy* of a finite measurable partition and its *conditional entropy* with respect to another finite measurable partition.

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $\xi, \eta \subset \mathcal{B}$ be finite partitions. Prove that $H_\mu(\xi|\eta) \leqslant H_\mu(\xi)$.

Prove that $H_\mu(T^{-1}\xi) = H_\mu(\xi)$.

Define $h_\mu(T, \xi)$ and $h_\mu(T)$. Prove that the limit in the definition exists.

[*You may use without proof the properties of subadditive sequences.*]

Prove that

$$h_\mu(T, \xi) = \inf_F \frac{1}{|F|} H_\mu\Big(\bigvee_{n \in F} T^{-n}\xi \Big),$$

where F runs through all finite subsets of $\mathbb{Z}_{\geqslant 0}$.

[*In this last part of the question, you may use without proof any results of the course.*]

4

Define the *K-mixing* property of a measure preserving system and prove that it implies *mixing*.

Define the *tail σ-algebra* of a finite measurable partition in a measure preserving system.

Prove that a system is K-mixing if and only if the tail σ-algebra of every finite measurable partition is trivial.

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. We write $\mathcal{P}$ for the collection of sets $A \in \mathcal{B}$ such that $h_\mu(T, \xi_A) = 0$, where $\xi_A = \{A, X \backslash A\}$. Prove that $\mathcal{P}$ is a σ-algebra. Prove that $A \in \mathcal{P}$ if and only if there is a finite partition $\xi \subset \mathcal{B}$ and a set $A' \in \mathcal{T}(\xi)$ such that $\mu(A \triangle A') = 0$.

[*In this last part of the question, you may use without proof any results of the course.*]

END OF PAPER