MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2019 1:30 pm to 4:30 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider Ω a connected open bounded set of $\mathbb{R}^d$ and let $x_0 \in \Omega$ and let $r > 0$ be such that $\overline{B(x_0;r)} \subset \Omega$. We denote by dS the surface measure on $\partial B(x_0;r)$ and we denote ω_d the Lebesgue measure of the unit ball, i.e., $\omega_d = \text{meas}(B(0;1))$. Recall that if $u \in C^2(\Omega)$ is harmonic in Ω then

$$u(x_0) = \frac{1}{\omega_d r^d} \int_{B(x_0;r)} u(x)dx = \frac{1}{d\omega_d r^{d-1}} \int_{\partial B(x_0;r)} u(x)dS, \qquad \text{for any } \overline{B(x_0;r)} \subset \Omega. \quad (1)$$

(i) Let $u \in C^0(\Omega)$ satsfying (1) in Ω such that there exists $y \in \Omega$ with $u(y) = \sup_\Omega u$ or $u(y) = \inf_\Omega u$. Prove that u must be constant in Ω. *Hint : Consider the set* $\Omega_M := \{x \in \Omega;\, u(x) = M\}$ *for* $M := \sup_\Omega u$ *and prove that* $\Omega_M = \Omega$.

(ii) Prove that any continuous function in Ω satisfying (1) for any x_0, r such that $\overline{B(x_0;r)} \subset \Omega$ must be harmonic in Ω. *Hint : Combine point (i) with the solvability of the Dirichlet problem on balls proved in lectures, i.e., consider the unique harmonic function* h *in* $B(x_0;r)$ *such that* $h = u$ *on* $\partial B(x_0;r)$.

(iii) Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a radial function such that

$$\text{supp}(\varphi) \subset B(0,1), \qquad \int_{\mathbb{R}^d} \varphi(x)udx = \omega_d \int_0^1 \varphi(r)r^{d-1}dr = 1.$$

and let

$$\varphi_\epsilon(z) = \epsilon^{-d}\varphi\left(\frac{z}{\epsilon}\right), \qquad \forall z \in \mathbb{R}^d, \quad \forall \epsilon > 0.$$

Denoting by $*$ the convolution of functions in $\mathbb{R}^d$, i.e.,

$$u * \varphi_\epsilon(x) = \int_{\mathbb{R}^d} u(y)\varphi_\epsilon(x-y)dy.$$

Prove that, for every $x \in \Omega$,

$$u * \varphi_\epsilon(x) = \int_0^1 \int_{\mathbb{S}^{d-1}} u(x + \epsilon r\omega)\varphi(r\omega)r^{d-1}drd\omega, \quad \epsilon < \text{dist}(x, \partial\Omega). \qquad (2)$$

Hint: Observe that, as φ *is a radial function, one can write* $\varphi_\epsilon(y-x) = \varphi_\epsilon(x-y)$. *Then, use the change of variables* $y \mapsto x + y$ *and later spherical coordinates* $y = r\omega$, *for* $r \in (0, +\infty)$ *and* $\omega \in \mathbb{S}^{d-1}$.

(iv) If u is a continuous function satisfying (1), use (2) and the properties of φ_ϵ, to prove that

$$u(x) = \varphi_\epsilon * u(x), \qquad \forall x \in \Omega, \quad \epsilon < \text{dist}(x, \partial\Omega). \qquad (3)$$

Deduce from (3) that $u \in C^\infty(\Omega)$.

(v) Suppose that $\Omega' \subset \Omega$ is an open set whose closure $\overline{\Omega'}$ is such that $R(\Omega') := \text{dist}(\Omega', \partial\Omega) > 0$. Using (1), show that for any u harmonic function $u \in C^\infty(\Omega)$,

$$\max_{j=1,\dots,d} \max_{\Omega'} |\partial_{x_j} u| \leqslant \frac{d}{R(\Omega')} \max_{\Omega} |u|. \qquad (4)$$

Hint: Recall that meas$(\partial B(0,R)) = d\omega_d R^{d-1}$.

2 Let Ω be a connected bounded open set of $\mathbb{R}^d$ with boundary $\partial\Omega$ and let $x_0 \in \partial\Omega$.

- The point x_0 is said to be regular with respect to the Laplacian if there exists a continuous function $w \in C^0(\overline{\Omega};\mathbb{R})$ satisfying the properties

$$w(x_0) = 0, \quad \Delta w \leqslant 0 \text{ in } \Omega, \quad w > 0 \text{ in } \overline{\Omega} \setminus \{x_0\}\,. \tag{1}$$

- $\partial\Omega$ satisfies the exterior sphere condition at x_0 if there exists a ball $B(y,R)$ such that $\overline{B(y,R)} \cap \overline{\Omega} = \{x_0\}$.

(i) ($d = 2$) Prove that if $\Omega \subset \mathbb{R}^2$ satisfies the exterior sphere condition at a point $x_0 \in \partial\Omega$, then x_0 is a regular point for the Laplacian.
Hint: Consider, for a given $z \in \mathbb{R}^2$, a function of the form $w(x) = \log(\delta|x-y|)$, for $\delta \in \mathbb{R}$ to be chosen.

(ii) Consider an operator of the form

$$\mathcal{L} = \sum_{i,j=1}^{d} a^{ij}(x)\partial^2_{x_i x_j} + \sum_{j=1}^{d} b_j(x)\partial_{x_j} + c(x), \quad x \in \Omega, \tag{2}$$

with smooth coefficients and satisfying,

$$a^{ij}(x) = a^{ji}(x), \tag{3}$$

$$\exists\lambda > 0 \text{ such that } \sum_{i,j} a^{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^d, \tag{4}$$

$$\exists\ell > 0 \text{ such that } \max_{j=1,\dots,d} |b_j(x)| < \lambda\ell. \tag{5}$$

where λ and ℓ are two given real constants. State and prove the *weak maximum principle* for this operator under the assumptions (3), (4), (5) and that $c = 0$.

(iii) If we assume that $c > 0$, does the weak maximum principle hold true? Find a proof or a counterexample according to your answer.

(iv) Suppose $c = 0$. Let $\varphi \in C^0(\partial\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ be fixed, for some $\alpha \in (0,1)$. Use the previous point to prove that solutions to the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

are unique in $C^0(\overline{\Omega}) \cap C^2(\Omega)$.

(v) Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be the unique solution of (6) as in the previous point. Does $u \in C^{2,\alpha}(\Omega)$ hold? Give sufficient conditions on φ, f and on the regularity of $\partial\Omega$ to guarantee that $C^{2,\alpha}(\overline{\Omega})$.

3 Let $d = 3$ and $B(x_0, r) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; \quad |x - x_0| < r\}$ for any $r > 0$ and $x_0 \in \mathbb{R}^3$. Set in particular $B = B(0, 1)$. Consider $\mathcal{L} = \sum_{i,j=1}^{3} \partial_{x_j} \left(a^{ij}(x)\partial_{x_i}\right)$ for the matrix

$$A(x) = (a^{ij}(x)) = \begin{pmatrix} 1 - \frac{x_1^2}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \forall x \in B. \tag{1}$$

Given $f \in L^q(B)$ with $q \in (1, \infty)$, consider the Dirichlet problem

$$-\mathcal{L}u = f \quad \text{in } D, \qquad\qquad u = 0 \quad \text{on } \partial D. \tag{2}$$

(i) Fix $r \in (0, 1)$ and $x_0 \in B$ such that $\overline{B(x_0, r)} \subset B$. Recall that a weak solution $u \in H^1(B)$ of (2) satisfies

$$\int_{B(x_0,r)} \left(\nabla u \cdot \nabla \phi - \frac{x_1^2}{2}\partial_{x_1} u \partial_{x_1} \phi\right) dx = \int_{B(x_0,r)} f\phi dx, \qquad \forall \phi \in H_0^1(B(x_0, r)). \tag{3}$$

Consider $w \in H_0^1(B(x_0, r))$ satisfying $\Delta w = 0$ weakly in $B(x_0, r)$, i.e.,

$$\int_{B(x_0,r)} \nabla w \cdot \nabla \phi dx = 0, \qquad \forall \phi \in H_0^1(B(x_0, r)). \tag{4}$$

Defining $v = u - w$, prove that $v \in H_0^1(B(x_0, r))$ satisfies

$$\int_{B(x_0,r)} |\nabla v|^2 dx \leqslant C_1 \int_{B(x_0,r)} |\nabla u|^2 dx + C_2 \left(\int_{B(x_0,r)} |f|^{\frac{6}{5}} dx\right)^{\frac{5}{3}}, \tag{5}$$

for some constants $C_1, C_2 > 0$.

(ii) Prove that, if $q = 2$, then for some $C_3 > 0$

$$\left(\int_{B(x_0,r)} |f|^{\frac{6}{5}} dx\right)^{\frac{5}{3}} \leqslant C_3 \left(\int_{B(x_0,r)} |f|^2 dx\right) r^{1+2\alpha}, \qquad \text{for } \alpha = \frac{1}{2}. \tag{6}$$

(iii) Recall that a result from lectures guarantees that if $g \in H^1_{loc}(B)$ satisfies

$$\int_{B(x_0,\rho)} \left| g(x) - \frac{1}{\text{meas}(B(x_0, \rho))} \int_{B(x_0,\rho)} g(y) dy \right|^2 dx \leqslant M^2 \rho^{3+2\alpha}, \quad \forall B(x_0, \rho) \subset B, \tag{7}$$

then $g \in C^{0,\alpha}(B)$. Assume that there exists $C_4 > 0$ such that

$$\int_{B(x_0,\rho)} |\nabla u|^2 dx \leqslant C_4 \rho^{1+2\alpha} \left(\|\nabla u\|^2_{L^2(B)} + \|f\|^2_{L^2(B)}\right), \quad \forall B(x_0, \rho) \subset B. \tag{8}$$

Prove that $u \in C^{0,\alpha}(B)$.

4 Let $\Omega \subset \mathbb{R}^d$ be a connected open bounded set with boundary $\partial\Omega$ of class $C^{2,\alpha}$, for some $\alpha \in (0,1)$. Let $F \in C^\infty(\mathbb{R})$ be non-negative, with bounded derivatives and satisfying

$$|z| \leqslant F(z) \leqslant 2|z|, \quad \forall z \in \mathbb{R}, \tag{1}$$

$$\|F(g)\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant \|g\|^2_{C^0(\overline{\Omega})}, \quad \forall g \in C^{2,\alpha}(\overline{\Omega}). \tag{2}$$

Given $f \in C^{0,\alpha}(\overline{\Omega})$ with $f \leqslant 0$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ with $\varphi \geqslant 0$, consider the semilinear Dirichlet problem

$$-\Delta u + F(u) = f \quad \text{in } \Omega, \qquad\qquad u = \varphi \quad \text{on } \partial\Omega. \tag{3}$$

Consider, for $\delta > 0$,

$$X_\delta := \left\{ w \in C^{2,\alpha}(\overline{\Omega}); \quad \|w\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant \delta \right\},$$

which is a convex closed subset of the Banach space $C^{2,\alpha}(\overline{\Omega}) \subset C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

(i) Let $0 < \delta < 1$. For any $w \in X_\delta$ consider the linear Dirichlet problem

$$-\Delta u + F(w) = f \quad \text{in } \Omega, \qquad\qquad u = \varphi \quad \text{on } \partial\Omega. \tag{4}$$

Using Schauder theory from lectures, prove that the mapping

$$T : X_\delta \mapsto C^{2,\alpha}(\overline{\Omega}); \quad T(w) = u$$

is well defined. Derive an estimate relating $\|T(w)\|_{C^{2,\alpha}}$ to the norms $\|F(w)\|_{C^{0,\alpha}}$, $\|T(w)\|_{C^0}$, $\|f\|_{C^{0,\alpha}}$ and $\|\varphi\|_{C^{0,\alpha}}$.

(ii) Assuming that

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant \epsilon, \tag{5}$$

prove that, if $\delta < 1$ is chosen small enough, there exists $\epsilon = \epsilon(\delta) > 0$ small enough such that $T(X_\delta) \subseteq X_\delta$.

(iii) Is T always a contraction in X_δ for $\delta > 0$ small enough? (*Hint: Consider examples in dimension one that can be explicitely solved.*)

END OF PAPER