UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2019 9:00 am to 12:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Analysis of Partial Differential Equations

Denote by $Q := (0, L)^n$ the open cube of side length L in $\mathbb{R}^n$, and let $U \subset \mathbb{R}^n$ be a connected, bounded open set with C^∞ boundary.

a) Show that for any $u \in H^1(\mathbb{R}^n)$, the estimate:

$$\|u\|^2_{L^2(Q)} \leqslant |Q| \overline{u}^2_Q + \frac{n}{2} L^2 \|Du\|^2_{L^2(Q)},$$

holds, where for any open set $W \subset \mathbb{R}^n$ and integrable function $w : W \to \mathbb{R}$, we define:

$$\overline{w}_W := \frac{1}{|W|} \int_W w(x) dx,$$

to be the mean of w over W.

b) Suppose that $(u_i)_{i=1}^\infty$ is a sequence of functions $u_i \in H^1(U)$.

i) State what it means for $(u_i)_{i=1}^\infty$ to converge weakly in $H^1(U)$ to some $u \in H^1(U)$.

ii) Show that if $(u_i)_{i=1}^\infty$ is bounded in $H^1(U)$, then there exists a subsequence $(u_{i_j})_{j=1}^\infty$ and $u \in H^1(U)$ such that $u_{i_j} \to u$ strongly in $L^2(U)$.

c) Show that there exist constants C_1, C_2, depending only on U, such that the inequalities:

i) $\|u - \overline{u}_U\|_{L^2(U)} \leqslant C_1 \|Du\|_{L^2(U)}$,

ii) $\|u\|_{L^2(U)} \leqslant C_2 \left(\|Du\|_{L^2(U)} + \|u\|_{L^2(\partial U)} \right)$,

hold for all $u \in H^1(U)$.
[Hint: suppose that the result is false, and derive a contradiction.]

You may assume the Sobolev approximation and extension theorems, together with standard results concerning Hilbert spaces.

2 Analysis of Partial Differential Equations

a) State without proof the Lax–Milgram theorem for a bilinear form $B : H \times H \to \mathbb{R}$, where H is a real Hilbert space.

b) Let $U \subset \mathbb{R}^n$ be open and bounded, with C^∞ boundary. Consider the system of elliptic equations:

$$\left.\begin{aligned} -\Delta u + u + w &= f \\ -\Delta w + w - 3u &= g \end{aligned}\right\} \quad \text{in } U, \tag{1}$$

where $u, w : U \to \mathbb{R}$ are the unknowns, $f, g \in L^2(U)$ are given functions and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator. We suppose that u, w are subject to the boundary conditions:

$$\left.\begin{aligned} u &= 0 \\ \frac{\partial w}{\partial \nu} &= 0 \end{aligned}\right\} \quad \text{on } \partial U, \tag{2}$$

with $\frac{\partial w}{\partial \nu} = \sum_{i=1}^n \nu_i \frac{\partial w}{\partial x_i}$, where ν is the outward unit normal to ∂U.

i) Define a *weak solution* to the system of equations (1) subject to boundary conditions (2), clearly identifying the spaces to which u, w belong. Show that if a weak solution is such that $u, w \in C^2(\overline{U})$, then the equations (1), (2) hold classically.

ii) Show that there exists a unique weak solution to the problem for any $f, g \in L^2(U)$.

3 Analysis of Partial Differential Equations

a) State what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be *real analytic* at a point $y \in \mathbb{R}^n$.

b) Suppose $a^{ij}, b^i, c : \mathbb{R}^n \to \mathbb{R}$ are real analytic functions for $i, j = 1, \ldots, n$ and consider the second order linear differential equation:

$$Lu := \sum_{i,j=1}^{n} a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i \frac{\partial u}{\partial x^i} + cu = 0. \tag{1}$$

i) State what it means for a real analytic surface $\Sigma \subset \mathbb{R}^n$ to be *characteristic* for the equation (1).

ii) Suppose Σ is not characteristic at $p \in \Sigma$. What data must be prescribed on Σ such that a unique real analytic solution to (1) exists in a neighbourhood of p?

c) Find and sketch all of the characteristic surfaces of the following equation defined on $\mathbb{R}^2$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y} \left[\left(1 - y^2\right)^2 \frac{\partial u}{\partial y} \right]. \tag{2}$$

Show that if $u(0, y)$ and $u_x(0, y)$ are given real analytic functions of y, then a unique real analytic solution exists in a neighbourhood of $\{x = 0\}$.

Note that in view of the dimension, the characteristic surfaces will in fact be curves. You may assume without proof the Cauchy–Kovalevskaya theorem and its corollaries.

d) Show that if $u \in C^\infty(\mathbb{R}^2)$ satisfies (2), then:

$$\frac{d}{dx} \int_{-1}^{1} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(1 - y^2\right)^2 \left(\frac{\partial u}{\partial y} \right)^2 \right] dy = 0.$$

Deduce that if $u(0, y) = u_x(0, y) = 0$ for $y \in (-1, 1)$, then $u(x, y) = 0$ for $(x, y) \in \mathbb{R} \times (-1, 1)$.

e) For $\epsilon > 0$, let $I_\epsilon = (-1 + \epsilon, 1 - \epsilon)$. Suppose that $\phi, \psi \in C_c^\infty(I_\epsilon)$. Find and sketch the maximal open set $D_\epsilon \subset \mathbb{R}^2$ for which there exists a *unique* $u \in C^\infty(D_\epsilon)$ solving (2) with $u(0, y) = \phi(y)$, $u_x(0, y) = \psi(y)$ for $y \in I_\epsilon$.

You may assume results from lectures concerning hyperbolic equations.

4 Analysis of Partial Differential Equations

For $r > 0$ and $x_0 \in \mathbb{R}^n$ let $B_r(x_0) \subset \mathbb{R}^n$ be the open ball of radius r centred at x_0, i.e. $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

a) Let A be a constant, symmetric matrix, with components A^{ij}, and suppose there exists $\theta > 0$ such that $\sum_{i,j=1}^n A^{ij}\xi_i\xi_j \geqslant \theta|\xi|^2$ for all $\xi \in \mathbb{R}^n$. Define:

$$L_0 u = \sum_{i,j=1}^n A^{ij} D_i D_j u.$$

Show that if $u \in C_c^\infty(B_r(x_0))$ for some $r > 0$, $x_0 \in \mathbb{R}^n$ then:

$$\theta \|D^2 u\|_{L^2(B_r(x_0))} \leqslant \|L_0 u\|_{L^2(B_r(x_0))}.$$

b) Suppose that $a^{ij} = a^{ji} \in C^0(\mathbb{R}^n)$, and define:

$$Lu = \sum_{i,j=1}^n a^{ij} D_i D_j u.$$

Fix $r > 0, x_0 \in \mathbb{R}^n$. Show that there exists $\epsilon > 0$ depending only on θ, n such that if $\|a^{ij} - A^{ij}\|_{L^\infty(B_r(x_0))} < \epsilon$ for all $i, j = 1, \ldots, n$, then

$$\frac{\theta}{2}\|D^2 u\|_{L^2(B_r(x_0))} \leqslant \|Lu\|_{L^2(B_r(x_0))}$$

holds for all $u \in C_c^\infty(B_r(x_0))$.

[Hint: you may wish to consider the identity $Lu = L_0 u + \sum_{i,j=1}^n (a^{ij} - A^{ij}) D_i D_j u$]

c) Suppose now that $a^{ij} \in C^0(\mathbb{R}^n)$ satisfy the uniform ellipticity condition:

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2 \qquad \text{for all } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n.$$

Show that if $U, W \subset \mathbb{R}^n$ are open sets with $W \subset\subset U$, then there exists a constant C depending on a^{ij}, U, W such that:

$$\|D^2 u\|_{L^2(W)} \leqslant C \left(\|Lu\|_{L^2(U)} + \|u\|_{H^1(U)}\right).$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

[Hint: You may wish to apply your result from part c) with $A^{ij} = a^{ij}(x_0)$ and r chosen appropriately. You may assume any results you require concerning partitions of unity.]

d) Deduce that if $u \in H^1(U)$ satisfies $Lu \in L^2(U)$, then in fact $u \in H^2_{loc.}(U)$.

END OF PAPER