MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2019 9:00 am to 12:00 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

Questions 1 and 4 are each worth 18 points.
Question 2 is worth 20 points.
Questions 3 and 5 are each worth 22 points.

All Lie algebras on this exam are assumed to be finite dimensional over $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
Triangular graph paper
(types A_2, B_2 and G_2)

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what it means for a subspace I of a Lie algebra $\mathfrak{g}$ to be an *ideal* of $\mathfrak{g}$. Prove that every term $\mathfrak{g}^{(n)}$ in the derived series is an ideal of $\mathfrak{g}$. [Prove any result you use.]

(b) Define what it means for a Lie algebra to be *simple*. Prove that $\mathfrak{sl}_2(\mathbb{C})$ is simple. [Prove any result you use.]

(c) Define the *Killing form* κ of a Lie algebra $\mathfrak{g}$. Show that if $\mathfrak{g}$ is simple, then its Killing form is nondegenerate. [You may use without proof the fact that that if $\kappa(x, y) = 0$ for all $x, y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable. Prove any other result that you use.]

2

Let

$$\mathfrak{g} = \mathfrak{sp}_6(\mathbb{C}) = \{x \in \mathfrak{gl}_6(\mathbb{C}) \mid Jx + x^T J = 0\}$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{t}$ be the space of diagonal matrices in $\mathfrak{g}$, and let Φ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. [For parts (a) - (c), you do not need to provide proofs for your answers.]

(a) Explicitly describe the elements of Φ as maps $\mathfrak{t} \to \mathbb{C}$.

(b) Identify a root basis $\Delta \subset \Phi$. Draw the Dynkin diagram of Φ and label it with the elements of Δ.

(c) For each element $\alpha \in \Delta$, explicitly describe the image of the elements of Δ under the simple reflection w_α.

(d) Let $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$ be the dual root system. Prove that $\check{\Delta} := \{\check{\alpha} \mid \alpha \in \check{\Delta}\}$ forms a root basis of $\check{\Phi}$. Draw the Dynkin diagram of $\check{\Phi}$ and label it with the elements of $\check{\Delta}$. [You do not need to prove that $\check{\Phi}$ forms a root system.]

3

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank ℓ over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$ and corresponding root system Φ. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$ be a choice of root basis for Φ. Recall that an element $x \in \mathfrak{g}$ is called *regular* if the centralizer $\mathfrak{z}_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [xy] = 0\}$ has dimension ℓ. [In this problem, you may use any result from the course if clearly stated.]

(a) State and prove a criterion in terms of roots for an element $t \in \mathfrak{t}$ to be regular.

(b) Show that if $\ell > 1$ and $x \in \mathfrak{g}_\alpha$ for some root α, then x is not regular.

(c) Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Suppose

$$\psi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$$

is an injective homomorphism such that $\psi(h) \in \mathfrak{t}$ and $\psi(e) = \sum_{i=1}^{\ell} e_{\alpha_i}$ for some nonzero elements $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$. Show that $\psi(h)$ and $\psi(e)$ are regular. [You do not need to prove the existence of ψ.]

4

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$ and corresponding root system Φ. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$ be a choice of root basis, and let $\{\omega_1, ..., \omega_\ell\}$ be the fundamental weights with respect to this choice of Δ. Given a dominant weight λ, let $V(\lambda)$ be the irreducible representation of $\mathfrak{g}$ with highest weight λ.

(a) State the Weyl dimension formula, briefly defining the notation you use.

For the rest of the problem, assume $\mathfrak{g} = \mathfrak{so}_5(\mathbb{C})$ and that α_1 is a short root.

(b) Let $\lambda = a\omega_1 + b\omega_2$ be a dominant weight. Using the Weyl dimension formula, find a formula for $\dim V(\lambda)$ in terms of a and b. [You do not need to prove the Weyl dimension formula.]

(c) Let V be the defining 5-dimensional representation of $\mathfrak{g}$. Decompose $V \otimes V$ into irreducible subrepresentations, i.e. find $\lambda_1, \lambda_2, ..., \lambda_n$ such that $V \otimes V \simeq V(\lambda_1) \oplus ... \oplus V(\lambda_n)$ as a representation of $\mathfrak{g}$. Explain your logic.

(d) Let $M(\omega_2)$ be the Verma module of highest weight ω_2. Describe the weights of $M(\omega_2)$ and the weights of its maximal proper submodule in terms of α_1 and α_2. [You do not need to provide a proof for your answer to part (d).]

5

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t}$ and root system Φ. Let $\Phi_0 \subset \Phi$ be the subset of roots of maximal length. Let

$$\mathfrak{h} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha.$$

Let Δ be a root basis of Φ. [Throughout this problem, you may use any result from the course.]

(a) Show that there exists a unique choice of simple roots Δ_0 for Φ_0 such that the fundamental Weyl chamber for Δ_0 contains the fundamental Weyl chamber for Δ. [You may use without proof that Φ_0 is a root system of rank ℓ.]

For the rest of the problem, assume that $\mathfrak{g}$ is of type G_2, and fix the root basis Δ_0 as in part (a). You may use without proof that Φ_0 is a root system of type A_2 and that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_3$. Let $\{\omega_1, \omega_2\}$ be the fundamental weights for Φ_0 corresponding to Δ_0. For a dominant weight λ, let $V(\lambda)$ be the irreducible representation of $\mathfrak{h}$ of highest weight λ.

(b) Suppose $\lambda = a\omega_1 + b\omega_2$ is a dominant weight for Φ_0 with respect to Δ_0. State and prove a criterion on the pair (a, b) for λ to be a dominant weight for Φ with respect to Δ.

(c) Let V be the 7-dimensional irreducible representation of $\mathfrak{g}$. Decompose V under its restriction to $\mathfrak{h}$, i.e. find weights $\lambda_1, ..., \lambda_n$ such that $V|_\mathfrak{h} \simeq V(\lambda_1) \oplus ... \oplus V(\lambda_n)$. Explain your logic.

(d) Suppose $\lambda = a\omega_1 + b\omega_2$ is a dominant weight for Φ with respect to Δ, and let U be the irreducible representation of $\mathfrak{g}$ with highest weight λ. Show that $U|_\mathfrak{h}$ contains a submodule isomorphic to $V(b\omega_1 + a\omega_2)$.

END OF PAPER