UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 9:00 am to 12:00 pm

PAPER 345

ENVIRONMENTAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a two-dimensional stratified flow above a deformable boundary. The fluid can be modelled as incompressible, inviscid and Boussinesq, and the domain assumed to be semi-infinite. Taking x in the horizontal direction and z vertically upwards, the deformable boundary is located at $z = \mathrm{Re}(\eta_0 e^{i(kx-\omega t)})$. Here, η_0 is the complex amplitude, $k > 0$ is the wavenumber and $\omega > 0$ is the frequency of the boundary oscillations. The motion of the boundary produces small perturbations to the fluid that is otherwise at rest with a continuous background stratification $\hat{\rho}(z)$.

(a) Under what conditions can the disturbance caused by the boundary be considered linear? For the case of a linear disturbance where the background stratification is characterised by a constant buoyancy frequency $N \geqslant 0$, derive the dispersion relation and determine the structure of the velocity $\mathbf{u} = (u, w)$ field perturbation. Describe the cases $\omega/N < 1$ and $\omega/N > 1$, sketching key features of the motion and the orientation of any wave velocities that may be relevant.

(b) Suppose now the stratification is given by

$$N(z) = \begin{cases} N_1, & 0 \leqslant z < H, \\ N_2, & z > H, \end{cases}$$

where $N_1 = 0$ and $N_2 > \omega$. What matching conditions are required at $z = H$? Determine the complex amplitude of the waves for $z > H$.

(c) Describe, with the aid of sketches, the flow field in the case where

$$N(z) = \begin{cases} N_1, & 0 \leqslant z < H, \\ N_2, & H < z < 2H, \\ N_1, & z > 2H, \end{cases}$$

for the same values of N_1 and N_2 as in (b). At what frequency (or frequencies) will the disturbance be maximum? You do not need to determine details of the solution, but you must justify your answer.

2

Consider a particle-laden layer of fluid extending along a horizontal boundary between barriers located at $x = 0$ and $x = L_0$. The particle volume concentration ϕ and layer depth h are given by $\phi = \phi_0 \ll 1$ and $h = H$ at $t = 0$. The particles have constant settling velocity W_s and density $\rho_p = \rho_0(1+\gamma)$, where γ is a constant with $0 < \gamma\phi_0 \ll 1$ and ρ_0 is the constant density of the fluid above the layer. There is a weak turbulent motion within the particle-laden layer such that the volume of the layer remains constant and the particles within the layer remain uniformly distributed in the vertical whilst still allowing particles to settle on the lower boundary. There is no resuspension of particles that have settled. The density of the fluid containing the particles is given by $\rho_f = \rho_0(1-\theta)$ with the heat from the particles causing the fluid expansion θ to vary as

$$\frac{d\theta}{dt} = \beta\phi,$$

where $\theta = 0$ at $t = 0$. Here, $\beta \geqslant 0$ is a constant and $\theta \ll 1$.

(a) Give a linearized equation of state for the bulk density ρ of the particle-laden layer and give an expression for the flux of particles onto the lower boundary. Determine the evolution of $\phi(t)$ and hence $\rho(t)$ of the layer. Show that the layer becomes statically unstable at $t = T_s$, where $T_s = \frac{H}{W_s}\ln(1 + \frac{\gamma W_s}{\beta H})$.

(b) Suppose now that the barrier at $x = L_0$ is removed at $t = 0$, allowing the layer to flow as a gravity current with depth h and velocity u. Derive the shallow water equations governing $h(x,t)$, $u(x,t)$, $\phi(x,t)$ and $\theta(x,t)$ behind the front of the current. Determine the characteristics and establish that the system is hyperbolic while the density field remains statically stable.

(c) By assuming h, ϕ and θ are constant along the length of the current, and specifying a constant Froude number for the front condition, derive an integral model for the advance of the current. Determine the evolution of the length $L(t)$ of the current in the limit $\beta > 0$, $W_s = 0$ and hence the maximum distance reached by the current. Determine also the maximum distance reached by the current in the limit $\beta = 0$, $W_s > 0$.

3

Consider a localised source emitting turbulent fluid at an angle θ_0 to the horizontal in the $x - z$ plane. The resulting three-dimensional 'forced plume' can be characterised by the mass flux Q, momentum flux $\mathbf{M} = (M_x, M_z)$ and buoyancy flux F. Here, M_x is the flux of horizontal momentum and M_z is the flux of vertical momentum. The conditions at the source are given by $Q = 0$, $\mathbf{M} = M_0(\cos\theta_0, \sin\theta_0)$ and $F = F_0$, and the density of the ambient fluid ρ_0 is constant. The resulting forced plume has an approximately circular cross-section that can be described by a top-hat profile of radius $b(s)$ with axial speed $V(s)$ and density $\rho(s)$, where s is the arc length from the source along the centreline of the forced plume and $\theta(s)$ is the local inclination of the centreline.

(a) Give expressions for $Q(s)$, $\mathbf{M}(s)$ and $F(s)$ in terms of b, V, ρ and θ for a Boussinesq flow. Describe the 'Batchelor entrainment' assumption with entrainment coefficient α and give an expression for the entrainment velocity u_e. Derive differential equations for the mass and buoyancy fluxes.

(b) In the limit $F_0 = 0$ for $\theta_0 = 0$, derive a differential equation for M_x. How does this change if $F_0 \neq 0$? Derive also, for $F_0 > 0$, a differential equation for M_z.

(c) Construct the 'jet length' L_J from the conditions at the source and describe briefly its relevance to the current problem. Determine the path taken by the centreline of the plume for $s \ll L_J$.

(d) For $\theta_0 = 0$ and $s \gg L_J$, the plume rises nearly vertically. Determine $Q(\hat{s})$, $M_z(\hat{s})$ and $F(\hat{s})$ as power laws in $\hat{s} = s + s_0$, where s_0 is a virtual origin. Describe the path of the plume centreline in this limit. Using physical arguments, comment on the location of the virtual origin relative to the position of the actual source. (You should provide an estimate of the position of the virtual origin based on scaling, but do not attempt a detailed calculation of the actual position.) Sketch the plume.

4

Consider an isolated particle of density ρ_p, with equivalent spherical diameter d, on a planar boundary in a fluid flow of density ρ_f.

(a) Perform a force balance in the downstream x-direction over a horizontal boundary to determine the condition for which the particle starts to move downstream. Take the granular friction coefficient as μ_s and neglect viscous effects. Show that the threshold Shields number can be written as

$$\Theta_{th,0} = \frac{4\mu_s}{3C_D},$$

with C_D the drag coefficient. Provide a sketch of this situation.

(b) Now suppose the boundary this grain rests on is inclined upwards at an angle α to the flow direction. Provide a sketch of this situation. Construct a modified force balance and determine the threshold Shields number $\Theta_{th,0}(\alpha)$.

(c) The equations relating sand flux q, saturated sand flux q_{sat}, shear stress τ, threshold shear stress τ_{th} and surface topography η are

$$\phi_b \frac{\partial \eta}{\partial t} = -\frac{\partial q}{\partial x},$$

$$L_{sat} \frac{\partial q}{\partial x} = q_{sat} - q,$$

$$q_{sat} = \phi_b \chi \left(\tau - \tau_{th}\right)^{\gamma},$$

with volume fraction of the packed bed ϕ_b and saturation length L_{sat}. Explain the meaning and origin of each of these equations. The local slope angle α is related to the height of the bedform η as

$$\tan(\alpha) = \frac{\partial \eta}{\partial x}.$$

Assuming the bedform as locally planar, derive linearized expressions for the perturbations $\hat{\eta}$, $\hat{q}$, $\hat{q}_{sat}$ and $\hat{\tau}$ for the case where $0 < \alpha \ll 1$ and $L_{sat} \neq 0$. Perform a linear stability analysis. Use your solution to part (b) to obtain an equation for $\hat{\tau}_{th}$ taking

$$\tau_{th} = \tau_{th,0} + \hat{\tau}_{th} \exp\left(\sigma t + ik(x - ct)\right),$$

with wavenumber k, velocity c and growth rate σ.

(d) Derive the dispersion relation connecting σ, k and c, and obtain expressions for $\sigma(k)$ and $c(k)$. Interpret the results and comment on the cases (i) $\tau_0 \approx \tau_{th,0}$ and (ii) $\tau_0 \gg \tau_{th,0}$.

END OF PAPER