UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 9:00 am to 11:00 am

PAPER 344

THEORETICAL PHYSICS OF SOFT CONDENSED MATTER

*Full marks can be achieved by complete answers to **TWO** questions.*

If you hand in more than two answers, all will be marked but the lowest mark is liable to be discounted.

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Answer all parts of the question.

(a) A classical degree of freedom $x(t)$ in contact with a heat bath obeys

$$-\frac{\delta A}{\delta x(t)} = -\zeta \dot{x} + f \tag{1}$$

where the classical action $A \equiv \int L(x, \dot{x}) dt$, ζ is a damping coefficient and f is Gaussian white noise of probability density $\mathbb{P}[f(t)] = \mathcal{N} \exp\left[-\frac{1}{2\sigma^2} \int f(t)^2 dt\right]$ with $\mathcal{N}$ a normalization constant. Assuming that $-\delta A/\delta x$ is invariant under time reversal, show that the probability density $\mathbb{P}_F[x(t)]$ for a trajectory $x(t)$ between $x_1(t_1)$ and $x_2(t_2)$, and that of the time-reversed trajectory $\mathbb{P}_B[x(t)]$, obey

$$\frac{\mathbb{P}_F}{\mathbb{P}_B} = \exp\left[\frac{2}{\sigma^2} \int_{t_1}^{t_2} \zeta \dot{x} \frac{\delta A}{\delta x(t)} \, dt\right].$$

(b) Show further that

$$-\int_{t_1}^{t_2} \dot{x} \frac{\delta A}{\delta x(t)} \, dt = H_2 - H_1$$

where H_2 and H_1 are the final and initial values of $H(x, \dot{x}) \equiv \dot{x} \frac{\partial L}{\partial \dot{x}} - L(x, \dot{x})$.

(c) Explain why microscopic reversibility of the underlying dynamics requires

$$\frac{\mathbb{P}_F}{\mathbb{P}_B} = \exp\left[-\beta(H_2 - H_1)\right] \tag{2}$$

with $\beta = 1/k_B T$, and hence establish that $\sigma^2 = 2k_B T \zeta$. State, and briefly explain, the generalization of (2) to the case where x represents a coarse-grained variable governed by a Helmholtz free energy F.

(d) A coarse-grained model is proposed for the evolution of the surface height $h(x)$ of a pile of sand in two dimensions subject to a random rain of grains from above:

$$\dot{h}_q(t) = -\alpha q^2 h_q + \eta_q(t) + \nu \delta_{q,0} \tag{3}$$

Here we have taken $x \in (0, L)$ and assumed $dh/dx = 0$ at the boundaries, so that $h(x) = \sum h_q \cos(qx)$ where $q = n\pi/L$ with $n \geqslant 0$ an integer. The parameter α quantifies rearrangement of particles across the surface, while ν quantifies the mean growth rate which we now set to zero by selecting a 'co-moving' frame. The Gaussian noise η obeys $\langle \eta_q(t) \eta_{q'}(t') \rangle = \sigma^2 \delta_{q,q'} \delta(t - t')$. By studying the time dependence of $\langle h_0(t)^2 \rangle$, show that this model does not admit a steady-state distribution $P_{ss}[h(x)]$ for the height variable, even in the co-moving frame, and hence can never reach Boltzmann equilibrium.

(e) Show nonetheless that for each mode of $q \neq 0$, (3) is isomorphic to (1) for the case of an overdamped particle ($L = -V$) in a harmonic potential. Hence, or otherwise, show that these modes achieve steady-state variances $\langle h_q^2 \rangle \propto q^{-2}$ at large enough times.

2

Answer all parts of the question.

For a polar liquid crystal with order parameter $\mathbf{p}(\mathbf{r}, t)$, the law of advection is

$$\frac{D\mathbf{p}}{Dt} = (\partial_t + \mathbf{v}.\nabla)\mathbf{p} + \mathbf{\Omega}.\mathbf{p} - \xi\mathbf{D}.\mathbf{p} \tag{1}$$

where $\mathbf{\Omega}$ and $\mathbf{D}$ are respectively the antisymmetric and symmetric parts of the velocity gradient tensor $\nabla_i v_j$. Here ξ is a material-dependent parameter.

(a) Without detailed calculation, explain why the coefficient of the $\mathbf{\Omega}$ term must be unity, as written, rather than a second material-dependent parameter.

(b) By considering the advective free energy increment in an incompressible displacement $\mathbf{u} = \mathbf{v}\Delta t$, show that the stress tensor $\Sigma^p_{ij}(\mathbf{r})$, caused by the polar ordering, obeys $\Sigma^p_{ij}(\mathbf{r}) = \Sigma^{(1)}_{ij} + \Sigma^{(2)}_{ij} + \Sigma^{(3)}_{ij}$ where

$$\begin{aligned} \nabla_i \Sigma^{(1)}_{ij} &= -p_i \nabla_j h_i \\ \Sigma^{(2)}_{ij} &= (p_i h_j - p_j h_i)/2 \\ \Sigma^{(3)}_{ij} &= \xi(p_i h_j + p_j h_i)/2 \end{aligned}$$

with $\mathbf{h}(\mathbf{r}) = \delta F/\delta \mathbf{p}(\mathbf{r})$ the molecular field.

(c) Consider a polar liquid crystal in two dimensions where $\mathbf{p} = (p_x, p_y) = (p\cos\theta, p\sin\theta)$ remains uniform in space, but evolves in time as a result of a steady shear flow, $\mathbf{v} = (gy, 0)$ where g is the shear rate. Assuming that the free energy density is

$$\mathbb{F} = \frac{a}{2}p^2 + \frac{b}{4}p^4 + \frac{\kappa}{2}(\nabla_i p_j)(\nabla_i p_j)$$

confirm that the molecular field $\mathbf{h}$ is colinear with $\mathbf{p}$ and find its magnitude.

Show that in the ordered phase ($p \neq 0$), the equation of motion $\frac{D\mathbf{p}}{Dt} = -\Gamma\mathbf{h}$ has a steady-state solution only for $|\xi| \geqslant 1$ and, even in that case, only for specific orientations of $\mathbf{p}$ obeying $\tan^2\theta = (\xi - 1)/(1 + \xi)$. Find the resulting magnitude of the polar order p as a function of a, b, Γ and g.

(d) By considering the special case $\xi = 0$, identify the likely behaviour of the system when $|\xi| < 1$.

3

Answer all parts of the question.

(a) The mean-field free energy density of a binary fluid mixture, with no symmetry relation between the two species, can be written as follows

$$f(\phi) = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \frac{c}{3}\phi^3 + d\phi \qquad (1)$$

with ϕ a compositional order parameter. Explain why the cubic term can be absorbed by shifts in the variable ϕ and in the parameters a and d. (You are not asked to calculate these shifts explicitly.) Explain further why the d term has no effect on phase equilibria. Hence show that the system has a critical point of the same character as in a symmetric binary fluid for which $c = d = 0$.

(b) Explain the physical character of the order parameter $\mathbf{Q}$ used to describe the transition from an isotropic, disordered phase to a nematic liquid crystal. Write down, to fourth order in $\mathbf{Q}$ the form of the mean field free energy $f(\mathbf{Q})$ that is the counterpart of (1). Explain why there is no term linear in $\mathbf{Q}$. Also explain the significance of the term cubic in $\mathbf{Q}$ and say why this term is absent in two dimensions.

(c) Assuming that Q_{ij} in three dimensions takes the form $\frac{3}{2}\lambda(n_i n_j - \delta_{ij}/3)$ with n_i a unit vector, show that $f(\mathbf{Q})$ can be written as

$$f(\lambda) = \bar{a}\lambda^2 + \bar{b}\lambda^4 + \bar{c}\lambda^3$$

and express $\bar{a}, \bar{b}, \bar{c}$ in terms of the parameters appearing in $f(\mathbf{Q})$. Show that the equilibrium value of λ in the ordered phase is positive for negative $\bar{c}$ and vice versa.

(d) Consider now the isotropic-nematic transition in an environment where rotational symmetry is broken by the presence of an external electric field $\mathbf{E}$. Show that for small fields the coupling term of lowest order in $\mathbf{E}, \mathbf{Q}$ that can appear in f is a term $-\chi E_i Q_{ij} E_j$. Show that for $\chi, \lambda > 0$ this term is minimized when $\mathbf{n}$ is aligned along $\pm\mathbf{E}$, and that with this choice the free energy $f(\lambda)$ acquires a linear term $\bar{d}\lambda$ with $\bar{d} < 0$.

(e) Explain why, for a non-conserved scalar order parameter such as λ, a continuous phase transition can arise only if there exists a pair of values $(\bar{a}, \lambda) = (\bar{a}_c, \lambda_c)$ at which $f' = f'' = f''' = 0$, where prime denotes a derivative with respect to λ. Show that in the absence of an external field these conditions only hold when $\bar{c} = 0$. Show also that with $\chi > 0$, for any given $\tilde{c} < 0$, a field strength $E = E_c$ can be found that allows the above conditions to be satisfied.

END OF PAPER