UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 1:30 pm to 4:30 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section A and **ONE** question from Section B.*

*There are **SEVEN** questions in total.*

Each question in Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) State the Dahlquist Equivalence Theorem and prove that the root condition is *necessary* for convergence.

(b) Given the multistep method determined by the polynomials

$$\rho(w) = w^3 - (1+2\alpha)w^2 + (1+2\alpha)w - 1, \qquad \sigma(w) = (\alpha-1)w^3 + 3(1-\alpha)w^2,$$

where α is a real parameter, determine the order of the method for different values of α and state for which α the method is convergent.

(c) Either find a value of α for which the method is both convergent and A-stable or prove that such an α does not exist.

2 Consider the three-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

(a) Determine the order of the method.
[Hint: You might identify the method with a collocation method.]

(b) Is the method A-stable?

(c) Is the method algebraically stable?

Carefully quote all the course material you are using throughout your answer.

3 The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0,1], \quad t \geqslant 0,$$

given with some initial conditions and homogeneous Dirichlet boundary conditions, is solved by the fully discretised method

$$(1+2\mu)u_m^{n+1} - \mu(u_{m-1}^{n+1} + u_{m+1}^{n+1}) = u_m^n, \qquad m = 1,\ldots,M, \quad n \geqslant 0,$$

where $u_m^n \approx u(m\Delta x, n\Delta t)$, $\Delta x = 1/(M+1)$ and $\mu = \Delta t/(\Delta x)^2$.

(a) Determine the order of the method.

(b) Fully justifying each step, determine the range of *Courant numbers* $\mu > 0$ for which the method is stable.

(c) Suppose that homogeneous Dirichlet boundary conditions are replaced by periodic boundary conditions. Modify the method to cater for this situation and determine for which $\mu > 0$ the new method is stable.

4 Consider the *linear Schrödinger equation*

$$i\frac{\partial u}{\partial t} = \Delta u - V(x,y)u, \qquad -1 \leqslant x, y \leqslant 1, \quad t \geqslant 0,$$

given in the square $[-1,1]^2$ with zero boundary conditions and an initial condition for $t = 0$. The *interaction potential* $V(x,y)$ is real.

(a) Prove that the exact solution of the equation conserves the L_2 norm, i.e. that

$$\|u(\,\cdot\,,t)\| = \sqrt{\int_{-1}^{1}\int_{-1}^{1} |u(x,y,t)|^2 dxdy}$$

is independent of $t \geqslant 0$.

(b) We discretise the Laplacian with the five-point formula, whereby producing a semi-discretised scheme,

$$iu'_{m,n} = \frac{1}{(\Delta x)^2}(u_{m-1,n}+u_{m+1,n}+u_{m,n-1}+u_{m,n+1}-4u_{m,n}) - V_{m,n}u_{m,n}, \quad m,n = 1,\ldots,M.$$

Here $\Delta x = 1/(M+1)$, $u_{m,n}(t) \approx u(m\Delta x, n\Delta x, t)$ and $V_{m,n} = V(m\Delta x, n\Delta x)$.

Prove that the semidiscretisation is stable.

5

Consider the two-point boundary value problem

$$-u'' + u = x, \qquad 0 \leqslant x \leqslant 1,$$

given with the zero boundary conditions $u(0) = u(1) = 0$.

(a) Carefully quoting all the relevant definitions and results, formulate the underlying variational problems and show that it has a unique minimum which is the weak solution of the ODE.

(b) We minimise the variational problem in an M-dimensional subspace $\mathcal{H}$ of $L_2[0,1]$ whose basis is $\{\varphi_1, \varphi_2, \ldots, \varphi_M\}$. All these basis elements are presumed to obey zero boundary conditions. Derive explicitly the underlying set of linear algebraic equations.

(c) We choose the φ_ks as hat functions on a uniform grid of size $h = 1/(M+1)$. Find the explicit form of the linear equations and prove that the underlying matrix is nonsingular.

SECTION B

6 Write an essay on Ritz and Galerkin finite element techniques, quoting all relevant definitions and theorems and accompanying your narrative with illustrative examples. There is no need to address (except for your illustrative examples) the construction of finite-element spaces.

7 Having first defined Runge–Kutta methods, write an essay on their linear and nonlinear stability. Accompany the essay with relevant examples. All theorems should be clearly formulated.

END OF PAPER