UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2018 9:00 am to 12:00 pm

PAPER 340

TOPICS IN MATHEMATICS OF INFORMATION

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What does it mean for a function $\psi \in L^2(\mathbb{R})$ to be an orthonormal wavelet of $L^2(\mathbb{R})$?

A multiresolution analysis (MRA) consists of a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, with $j \in \mathbb{Z}$, satisfying the following properties.

(I) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.

(II) For all $j \in \mathbb{Z}$, $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$.

(III) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(IV) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(V) There exists $\varphi \in V_0$ such that $\{\varphi(\cdot - k);\ k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

The function φ in (V) is called a scaling function for the MRA.

(b) Prove that $(I), (II)$ and (V) imply (III).

(c) Explain, without proof, how one can construct an orthonormal wavelet from an MRA.

(d) Is $\chi_{[-1/2,1/2)}$ a scaling function of an MRA? Justify your answer.

(e) One is often interested in wavelets which have a high number of vanishing moments and compact support. Explain why these two properties are desirable and any tradeoffs between the two properties. Give examples of two wavelets, and for your examples, explain under which circumstances one wavelet should be chosen over the other?

2

(a) Let $\varphi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ be a compactly supported scaling function of a Multiresolution Analysis (MRA). Show that its low pass filter m is necessarily a trigonometric polynomial which satisfies

(i) $|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1$ for all $\xi \in \mathbb{R}$,

(ii) $|m(0)| = 1$.

State clearly any properties of φ that you invoke.

(b) Let m be a trigonometric polynomial satisfying conditions (i) and (ii) listed in (a) and

(iii) $m(\xi) \neq 0$ for all $\xi \in [-\pi/2, \pi/2]$.

Assuming that $\Theta(\xi) := \prod_{j=1}^{\infty} m(\xi/2^j)$ converges uniformly on compact sets and

$$\frac{1}{2\pi} \int |\Theta(\xi)|^2 \, e^{-ik\xi} \mathrm{d}\xi = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0, \end{cases}$$

show that its inverse Fourier transform defines a scaling function of an MRA.

(c) In the construction of Daubechies wavelets of order $N \in \mathbb{N}$, the associated low pass filter is of the form

$$m(\xi) = \left(\frac{1 + e^{-i\xi}}{2} \right)^N L(\xi),$$

where L is a trigonometric polynomial of degree $N - 1$ such that $L(\pi) \neq 0$. What can you say about the vanishing moments of the associated wavelet? Justify your answer.

(d) Show that the scaling function φ associated to the low pass filter m in (c) has Fourier transform of the form

$$\hat{\varphi}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^N \prod_{j=1}^{\infty} L(\xi/2^j).$$

Moreover, show that $\sup_\xi |L(\xi)| < 2^{N-\alpha-1}$ implies that φ is uniformly Lipschitz-α. You may use without proof the fact that a function f is uniformly Lipschitz-α if there exist $C, \varepsilon > 0$ such that its Fourier transform satisfies $\left|\hat{f}(\xi)\right| \leqslant C/(1+|\xi|)^{\alpha+1+\varepsilon}$ for all $\xi \in \mathbb{R}$.

3

(a) Given an orthonormal basis $\mathcal{G} = \{g_m\}_{m\in\mathbb{N}}$ of a Hilbert space $\mathcal{H}$, what are the N-term linear f_N^l and nonlinear approximations f_N^n of an element $f \in \mathcal{H}$?

(b) Let $\mathcal{G}$ be an interval adapted orthonormal wavelet basis of $L^2[0,1]$ where the wavelet has q vanishing moments and is q-times continuously differentiable, and suppose that $f \in L^2[0,1]$ is piecewise polynomial of degree $p < q$. Show that the nonlinear approximation error is $\varepsilon^n(N,f) := \|f_N^n - f\|_{L^2}^2 = \mathcal{O}(\omega^N)$ for some $\omega \in (0,1)$ and that the linear approximation error $\varepsilon^l(N,f) := \left\|f_N^l - f\right\|_{L^2}^2 = \mathcal{O}(N^{-1})$.

(c) If $\mathcal{G}$ is an orthonormal Fourier basis of $L^2[0,1]$, what is the best linear and nonlinear approximation rate for functions defined on $[0,1]$ which are smooth except for a finite number of discontinuities?

(d) Given a 1D Multiresolution Analysis (MRA) for $L^2(\mathbb{R})$, explain how one can construct a wavelet basis of $L^2(\mathbb{R}^2)$.

(e) Suppose that $f \in L^2([0,1]^2)$ is piecewise polynomial of degree p except on a smooth curve of finite length. Consider its nonlinear approximation from the 2D wavelet basis of $L^2([0,1]^2)$, constructed from the 1D interval adapted wavelet basis of (b). Show that $\|f_N^n - f\|_{L^2}^2 = \mathcal{O}(N^{-1})$.

4 Let $x_0 \in \mathbb{R}^N$, $A \in \mathbb{R}^{m \times N}$ and let $y = Ax_0 + e$ where $e \in \mathbb{R}^m$ with $\|e\|_2 \leqslant \eta$ and $\eta \geqslant 0$. Consider the minimization problem

$$\min_{x \in \mathbb{R}^N} \|x\|_1 \text{ subject to } \|Ax - y\|_2 \leqslant \eta \tag{1}$$

(a) Let $x_0 \in \mathbb{R}^N$, $S = \mathrm{Supp}(x_0)$ and consider (1) with $\eta = 0$ and $y = Ax_0$. Show that x_0 is the unique solution to (1) if and only if A satisfies the null space property relative to the set S.

(b) Show that if there is a unique minimizer $\hat{x}$ to (1), then $\hat{x}$ must be m-sparse. (Hint: show that $\{a_j \,;\, j \in \mathrm{Supp}(\hat{x})\}$ where a_j denotes the j^{th} column of A, is a linearly independent set).

Let $s \in \mathbb{N}$. The sth restricted isometry constant $\delta_s(A)$ of a matrix $A \in \mathbb{C}^{m \times N}$ is the smallest value $\delta > 0$ such that

$$(1-\delta)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\delta)\|x\|_2^2 .$$

(c) Show that $|\langle Au,\, Av\rangle| \leqslant \delta_{s+t}(A)\,\|u\|_2\,\|v\|_2$ for all $u, v \in \mathbb{C}^N$ such that $\mathrm{Supp}(u) \cap \mathrm{Supp}(v) = \emptyset$ and $|\mathrm{Supp}(u)| = s$, $|\mathrm{Supp}(v)| = t$.

(d) Prove that if $\delta_s(A) < 1/3$, then any minimizer $\hat{x}$ to (1) satisfies

$$\|\hat{x} - x_0\|_2 \leqslant C\eta + \frac{\sigma_s(x_0)}{\sqrt{s}},$$

where $\sigma_s(x_0) = \min_{x \text{ is } s\text{-sparse}} \|x - x_0\|_1$ and C is a positive constant. You may use without proof any results concerning robust null space properties, provided that they are stated clearly.

5 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary.

(a) Define the total variation $|Df|(\Omega)$ of a function $f \in L^1(\Omega)$ and the space of functions of bounded variation.

(b) Consider the following minimization problem.

$$\min_{f \in L^2(\Omega)} |Df|(\Omega) + \frac{1}{2}\|f - g\|^2_{L^2(\Omega)}. \tag{1}$$

Prove that there exists a unique solution to this minimization problem. Let $b > a$ be real numbers. Prove that if $g(x) \in [a, b]$ for a.e. $x \in \Omega$, then we also have that the minimizer f_* is such that $f_*(x) \in [a, b]$ for a.e. $x \in \Omega$.

(c) State the proximal gradient descent algorithm and explain how this can be applied to solve a discretized version of (1), that is, letting $X = \mathbb{R}^{N \times N}$ for some $N \in \mathbb{N}$, consider the following minimization problem from some given $g \in X$:

$$\min_{u \in X}\{\|\nabla u\|_{2,1} + \frac{1}{2}\|u - g\|_2^2\} \tag{2}$$

where $\|q\|_2^2 = \sum_{i,j} q_{i,j}^2$ for $q \in X$, $\|p\|_{2,1} = \sum_{i,j}\sqrt{(p^1_{i,j})^2 + (p^2_{i,j})^2}$ for $p = (p^1, p^2) \in X \times X$, and $\nabla : X \to X \times X$ is the discrete gradient operator defined by

$$(\nabla u)_{i,j} = \begin{pmatrix} (D_x^+ u)_{i,j} \\ (D_y^+ u)_{i,j} \end{pmatrix},$$

with

$$(D_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & i < N \\ 0 & i = N, \end{cases} \qquad (D_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & j < N \\ 0 & j = N. \end{cases}$$

You may use, without proof, the dual formulation of (2).

(d) How can the proximal gradient descent algorithm be applied to solve

$$\min_{x \in \mathbb{R}^N} \|x\|_1 + \frac{1}{2}\|Ax - y\|_2^2,$$

where $A \in \mathbb{R}^{m \times N}$ and $y \in \mathbb{R}^m$?

6 Write an essay on image smoothing and edge enhancement with linear and nonlinear diffusion equations.

OR

Write an essay on the construction of wavelet bases for the unit interval $[0, 1]$.

END OF PAPER