UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 1:30 pm to 3:30 pm

PAPER 339

TOPICS IN CONVEX OPTIMISATION

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the linear program

$$\max_{x\in\mathbb{R}^n} \; r^T x \quad \text{subject to} \quad x \geqslant 0,\; \mathbf{1}^T x = 1 \tag{P}$$

where $r \in \mathbb{R}^n$ is given, and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ is the vector of all ones.

(a) Find analytically the optimal $x = x^*$ of (P). [5]

(b) We assume now that the vector r is "uncertain", i.e., we only know that it lies in a certain set

$$\mathcal{U} = \{r \in \mathbb{R}^n : \|P(r - r_0)\|_\infty \leqslant 1\},$$

where r_0 is a nominal value for r, P is a given $m \times n$ matrix, and $\|z\|_\infty = \max_j |z_j|$. Given this uncertainty we want to solve the following max-min problem, which is a *robust* counterpart of (P):

$$\max_{x\in\mathbb{R}^n} \left(\min_{r\in\mathcal{U}} r^T x \right) \quad \text{subject to} \quad x \geqslant 0,\; \mathbf{1}^T x = 1. \tag{R}$$

(i) Formulate the inner optimisation problem $\min_{r\in\mathcal{U}} r^T x$ (where x is fixed) as a linear program. [15]

(ii) Write the dual of this linear program and show that strong duality holds. [15]

(iii) Conclude that problem (R) is equivalent to the following *linear program*: [15]

$$\begin{array}{ll} \underset{x,\alpha,\beta}{\text{maximise}} & r_0^T x - (\alpha+\beta)^T \mathbf{1} \\ \text{subject to} & x \geqslant 0,\; \mathbf{1}^T x = 1 \\ & \alpha, \beta \geqslant 0, P^T(\alpha - \beta) = x. \end{array}$$

2 Consider the following optimisation problem:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximise}} & \|x\|_2^2 \\ \text{subject to} & |a_i^T x| \leqslant 1 \ \forall i = 1, \ldots, m, \end{array} \tag{P}$$

where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $\|x\|_2^2 = \sum_{i=1}^n x_i^2$. Let v^* be the optimal value of (P).

Consider the semidefinite relaxation:

$$\begin{array}{ll} \underset{X \in \mathbf{S}^n}{\text{maximise}} & \text{Tr}(X) \\ \text{subject to} & \text{Tr}(a_i a_i^T X) \leqslant 1 \ \forall i = 1, \ldots, m \\ & X \succeq 0. \end{array} \tag{SDP}$$

Let p^*_{SDP} be the optimal value of the SDP.

(a) Show that $p^*_{SDP} \geqslant v^*$. [5]

The purpose of the remaining questions is to prove the inequality:

$$v^* \geqslant \frac{1}{2\log(2m)} p^*_{SDP} \tag{1}$$

(b) Let X^* be the optimal solution of the SDP and let $X^* = V \Lambda V^T$ be an eigenvalue decomposition of X, where V is an orthogonal matrix $VV^T = V^T V = I_n$ and Λ is diagonal. For $\xi \in \{-1, 1\}^n$ define

$$\hat{x}(\xi) = V \Lambda^{1/2} \xi \quad \text{and} \quad x(\xi) = \frac{1}{\max_{i=1,\ldots,m} |a_i^T \hat{x}(\xi)|} \hat{x}(\xi).$$

Verify that $x(\xi)$ is feasible for (P) and that [15]

$$\|x(\xi)\|_2^2 = \text{Tr}(X^*) \frac{1}{(\max_{i=1,\ldots,m} |a_i^T \hat{x}(\xi)|)^2}.$$

(c) We want to show that there exists $\xi \in \{-1, 1\}^n$ such that

$$\left(\max_{i=1,\ldots,m} |a_i^T \hat{x}(\xi)| \right)^2 \leqslant 2\log(2m). \tag{2}$$

To do so we will use a probabilistic argument. You can use the following fact without proof:

Let $u_1, \ldots, u_m$ be vectors in $\mathbb{R}^n$ such that $\|u_i\|_2 \leqslant 1$ for all $i = 1, \ldots, m$. If $\xi \in \{-1, 1\}^n$ is uniformly distributed on $\{-1, 1\}^n$ then

$$\Pr_{\xi} \left[\max_{i=1,\ldots,m} |u_i^T \xi| \leqslant \alpha \right] > 1 - 2me^{-\alpha^2/2} \tag{3}$$

where $\Pr[A]$ denotes the probability of event A.

Using this result show that there is at least one $\xi \in \{-1, 1\}^n$ such that (2) holds. [20]

[*Hint: show that $u_i = (V\Lambda^{1/2})^T a_i$ have norm at most 1, and find α such that the right-hand side of* (3) *is nonnegative.*] Conclude that (1) holds. [10]

3

(a) Let f be a polynomial in one variable and assume there exist numbers $c_0, \ldots, c_n \geqslant 0$ such that

$$f(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}.$$

Show that $f(x) \geqslant 0$ for all $x \in [0,1]$. [5]

The purpose of the following questions is to prove a converse of part (a), under the assumption that f is strictly positive on $[0,1]$. This will allow us to define a convergent linear programming hierarchy for the optimization of polynomials on $[0,1]$. Given a function $f : [0,1] \to \mathbb{R}$ we define the Bernstein approximation:

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k} \quad \forall x \in [0,1].$$

We are going to assume the following important facts about B_n (we use the notation $\|f\| = \max_{x \in [0,1]} |f(x)|$, and $\mathbb{R}[x]_{\leqslant d}$ stands for polynomials of degree at most d):

(i) If f is continuous on $[0,1]$ then $\|B_n(f) - f\|_\infty \to 0$ as $n \to \infty$.

(ii) If $f \in \mathbb{R}[x]_{\leqslant d}$ then $B_n(f) \in \mathbb{R}[x]_{\leqslant d}$. Furthermore for $n \geqslant d$, B_n is invertible as a linear map on $\mathbb{R}[x]_{\leqslant d}$.

(b) Let $f \in \mathbb{R}[x]_{\leqslant d}$ be strictly positive on $[0,1]$, i.e., $f(x) > 0$ for all $x \in [0,1]$. Let $g_n = (B_n)^{-1}(f) \in \mathbb{R}[x]_{\leqslant d}$ for $n \geqslant d$. Show that for large enough n, $g_n \geqslant 0$ on $[0,1]$. [10]

(c) Show the following: if f is a polynomial that is strictly positive on $[0,1]$ then there exist $n \in \mathbb{N}$ and nonnegative coefficients $c_0, \ldots, c_n \geqslant 0$ such that [15]

$$f(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}.$$

(d) Using the previous question, design a *hierarchy* of linear programs to compute the minimum of a polynomial $f \in \mathbb{R}[x]$ on $[0,1]$. In other words, show that there is a sequence $v_1 \leqslant v_2 \leqslant \ldots \leqslant \min_{x \in [0,1]} f(x)$ with $v_n \to \min_{x \in [0,1]} f(x)$ as $n \to \infty$, such that v_n can be computed using a linear program with at most $n+1$ inequality constraints. [20]

END OF PAPER