UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 1:30 pm to 3:30 pm

PAPER 336

PERTURBATION METHODS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Use the method of steepest descents to find the leading-order term in the asymptotic expansion of

$$I = \int_C e^{-ik(z - 2e^{\pi i/4}\sqrt{z})} dz$$

for $k \in \mathbb{R}$, $k \gg 1$. The branch cut of $\sqrt{z}$ is taken along the negative imaginary axis, and C is a contour composed of $(-\infty, -\delta] \cup C_\delta \cup [\delta, \infty)$, with C_δ a semicircular contour of radius $0 < \delta \ll 1$ in the upper half plane so as to avoid the branch point. Identify both the saddle point, z_s, and the contour of steepest descents.

(b) Find the leading-order term in the asymptotic expansion of

$$J(z_0) = \int_C \frac{e^{-ik(z - 2e^{\pi i/4}\sqrt{z})}}{z - z_0} dz,$$

where z_0 is a complex constant with $|z_0| < 1$, and C the same contour as described in part (a). Be careful to consider different cases for z_0.

(c) Identify a distinguished limit, $z_0 \to z_d$, of $J(z_0)$, and show that when the limit is approached from below the steepest descents contour, then for $k \gg 1$

$$\lim_{z_0 \to z_d} J(z_0) \sim e^{-k} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}s^2}}{s - s_d} ds + 2\pi i \, e^{-ik(z_0 - 2e^{\pi i/4}\sqrt{z_0})},$$

where s_d is to be determined in terms of some or all of z_0, z_d, z_s. Show that this expression is consistent with your results from part (b).

2 The function $y(x)$ satisfies the differential equation

$$\varepsilon^2 y'' + 2(x-1)y' - 2\varepsilon y = \begin{cases} 2(x-1) & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{for } 1 \leqslant x \leqslant 2 \end{cases},$$

and boundary conditions

$$y(0) = a\,, \quad y(2) = b\,,$$

where $0 < \varepsilon \ll 1$, and a and b are order-one constants. State appropriate conditions on y and y' at $x = 1$.

By means of matched asymptotic expansions find the solution for $y(x)$ correct to and including $O(\varepsilon)$ terms for $0 \leqslant x \leqslant 2$. Briefly comment on the case when $b = a + 1$.

Suppose instead that $y(x)$ satisfies the differential equation (note the change of sign)

$$-\varepsilon^2 y'' + 2(x-1)y' - 2\varepsilon y = \begin{cases} 2(x-1) & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{for } 1 \leqslant x \leqslant 2 \end{cases},$$

with the same boundary conditions. *Without performing detailed calculations*, briefly outline the asymptotic structure of the solution.

Hints.

(i) Recall that

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad \text{and} \quad \text{erf}(\infty) = 1\,.$$

(ii) A particular solution for $Y(z)$ to

$$Y'' + 2zY' = 2a_1 z + a_2 + a_3 \text{erf}(z)\,,$$

where a_1, a_2 and a_3 are constants, is

$$Y = a_1 z + \int_0^z e^{-t^2} \int_0^t e^{u^2} \left(a_2 + a_3 \text{erf}(u)\right) du dt\,.$$

(iii) As $z \to \infty$

$$\int_0^z e^{-t^2} \int_0^t e^{u^2} du dt \to \tfrac{1}{2} \log|z| + C_1\,,$$

$$\int_0^z e^{-t^2} \int_0^t e^{u^2} \text{erf}(u) du dt \to \tfrac{1}{2} \log|z| + C_2\,,$$

where C_1 and C_2 are to be taken as known constants.

3 For $t \geqslant 0$, the function $y(t;\varepsilon)$ satisfies the differential equation

$$y_{tt} + \exp(-2\varepsilon t)(\varepsilon y_t + y) = 0\,,$$

and the initial conditions

$$y(0;\varepsilon) = 0\ , \quad y_t(0;\varepsilon) = 1\,.$$

Find the leading-order WKB solution for $y(t;\varepsilon)$.

Explain why the WKB solution is no longer valid when $t = O(\varepsilon^{-1}\ln(\varepsilon^{-1}))$, and find an asymptotic solution in this region by means of a shift in origin of t and a rescaling. Find the limiting behaviour of the solution for $t \gg \varepsilon^{-1}\ln(\varepsilon^{-1})$.

Hints.

(i) You may quote the exact solution to

$$y_{tt} + \exp(-2t)y = 0\,,$$

as

$$y = \alpha J_0(e^{-t}) + \beta Y_0(e^{-t})\,,$$

where J_0 and Y_0 are Bessel functions, and α and β are constants.

(ii) You may also quote the following limiting behaviours of $J_0(z)$ and $Y_0(z)$:

as $z \to 0$, $J_0(z) \sim 1 + \dots$ and $Y_0(z) \sim \dfrac{2}{\pi}\left(\ln\left(\tfrac{1}{2}z\right) + \gamma\right) + \dots\,,$

as $z \to \infty$, $J_0(z) \sim \left(\dfrac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \tfrac{\pi}{4}\right) + \dots$ and $Y_0(z) \sim \left(\dfrac{2}{\pi z}\right)^{\frac{1}{2}} \sin\left(z - \tfrac{\pi}{4}\right) + \dots\,,$

where γ is Euler's constant.

END OF PAPER