UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 9:00 am to 11:00 am

PAPER 335

DIRECT AND INVERSE SCATTERING OF WAVES

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a time-harmonic wave $\psi(x, y, z)$ with wave number $k = \omega/c$ propagating in a medium with refractive index $n(x, y, z) = c(x, y, z)/c_0$, where c_0 and c are the wave speed in free space and in the medium respectively, so $k = k_0 n$, where $k_0 = \omega/c_0$ is a reference wavenumber.

Assume that the plane wave components of $\psi(x, y, z)$ are propagating at small angles w.r.t. the horizontal x, therefore the reduced wave $E(x, y, z) = \psi(x, y, z)e^{-ik_0 x}$ is a slowly-varying function of x and obeys the parabolic equation.

The refractive index of the medium is given by

$$n(x, y, z) \;=\; 1 + \mu W(x, y, z) \tag{1}$$

where μ is a constant and $W(x, y, z)$ is the random part, which is normally distributed and statistically stationary, and has been normalised so that $< W >= 0$ and $< W^2 >= 1$

(i) Explain how equations for the moments of the field can be obtained by treating the scattering and diffraction separately. Then derive an equation of propagation for the first moment of the field, $< E(x, y, z) >$, and write the solution $< E(x, y, z) >$ at a generic point x in the medium.

(ii) Assume now that the medium is isotropic, and δ-correlated in the direction of propagation x:

$$< W(x, y_1, z_1)W(x, y_2, z_2) >= \delta(x_1 - x_2)B(\eta, \zeta) \;, \tag{2}$$

where B is a differentiable function of the distances $\eta = y_1 - y_2$ and $\zeta = z_1 - z_2$.

Express the solution $< E(x, y, z) >$ in terms of the power spectrum of the refractive index of the medium.

[You may use the following results:
If $f(\eta, \zeta)$ is isotropic, so $f(r, \theta) = f(r)$ in polar coordinates (r, θ), and

$$F(\nu_\eta, \nu_\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\eta, \zeta)e^{-i(\nu_\eta \eta + \nu_\zeta \zeta)} d\eta, d\zeta \tag{3}$$

then the following applies:

$$F(\nu_\eta, \nu_\zeta) = F(\nu) = \int_0^{\infty} f(r)J_0(\nu r)r dr \;, \quad \textit{where} \;\; \nu = |(\nu_\eta, \nu_\zeta)| \;, \tag{4}$$

together with the inverse transform

$$f(\nu) = \int_0^{\infty} F(\nu)J_0(\nu r)\nu d\nu \;. \tag{5}$$

Also, $J_0(0) = 1$.]

(iii) Write now an explicit expression for the above solution in the case when the medium has power spectrum given by:

$$S(\nu) = \mu^2 L^3 e^{-(\nu L)^2/4} \quad , \tag{6}$$

where L is the correlation length of the medium.

2 Consider a randomly inhomogeneous medium occupying the strip contained in a 3-dimensional space between $x = 0$ and $x = L$.
A time-harmonic acoustic source, for example a Gaussian beam centred at $\mathbf{z} = \mathbf{z}_0$, on the transverse, two-dimensional $\mathbf{z}$-plane, emits a signal at $x = 0$, in the direction parallel to the x-axis. An array of transducers of size A on the transverse plane at $x = L$ can both record and transmit acoustic signals.

(i) Describe what a time-reversal experiment is, and how it differs in a randomly inhomogeneous medium from the case of a deterministic medium. Give a brief, heuristic explanation of the differences.

Denote the initial time-harmonic source by $\Psi_0(x, \mathbf{z})$, and write a mathematical expression for the time-reversed, back-propagated field $\Psi_B(x, \mathbf{z})$ at $x = 0$, in the frequency domain.

(ii) Assume now that the initial source at $x = 0$ is a rapidly decaying function of $\mathbf{z}$, and consider the Wigner distribution of two vector fields on $\mathbb{R}^d$, $u(\mathbf{z})$ and $v(\mathbf{z})$, defined by:

$$W[u, v](\mathbf{z}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{z}'} u(\mathbf{z} + \frac{1}{2}\mathbf{z}') v^*(\mathbf{z} - \frac{1}{2}\mathbf{z}') d\mathbf{z}' \ . \tag{1}$$

By choosing the functions u and v appropriately, write the time-reversed, back-propagated field $\Psi_B(x, \mathbf{z})$ at $x = 0$ as an expression involving the integral of the Wigner distribution $W[u, v](\frac{\mathbf{z}+\mathbf{z}'}{2}, \mathbf{p})$, over the plane of the receiver:

$$W_A[u, v](\mathbf{z}, \mathbf{p}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\mathbf{p}\cdot\mathbf{z}'} W[u, v] \left(\frac{\mathbf{z} + \mathbf{z}'}{2}, \mathbf{p} \right) d\mathbf{z}' \ , \tag{2}$$

where $\mathbf{z}$, $\mathbf{z}'$ and $\mathbf{p}$ are 2-dimensional vectors.

(iii) Consider now the case where the medium occupying the strip between $x = 0$ and $x = L$ is free space containing random, well separated point scatterers, and the initial signal is ψ_0, emitted at the transverse plane at $x = 0$, then time-reversed and back-propagated at the transverse plane at $x = L$.

By writing the forward propagated wave formally as $\psi_R = H\psi_0$, relate the largest point scatterer (i.e. the one with largest reflected intensity) to the eigenvalues of the operator H^*H.

3 Consider a scalar wave $\psi_i(\mathbf{r})$ incident upon an inhomogeneity with known space-dependent refractive index $n(\mathbf{r}) = 1 + n_\epsilon(\mathbf{r})$, which occupies a volume $B \in \mathbb{R}^3$ and is embedded in free space.

(i) Express the scattered field $\psi_s(\mathbf{r})$ as an infinite series using the Born approximation.

Comment on the physical significance of the successive terms in the series, and give at least one condition for the validity of the first Born approximation.

(ii) Assuming that $n_\epsilon \ll 1$, write an approximate expression for ψ_s in the first Born approximation, disregarding terms of second order in n_ϵ.

Assume you are given a known incident field $\psi_{inc}(\mathbf{r})$ and a known, measured scattered field $\psi_s(\mathbf{r})$. By defining a suitable operator A, write this as an operator equation

$$A[n_\epsilon] = \psi_s \ , \tag{1}$$

and formulate the inverse problem of finding the unknown refractive index $n_\epsilon(\mathbf{r})$ as a Landweber iteration.

(iii) Derive a closed form expression for the n^{th} Landweber iterate, hence define a regularisation operator R_n, and show that the reciprocal of the iteration index in the Landweber approximation plays the part of a regularisation parameter. You may assume that $\psi_s \in D(A^\dagger)$ and $\| A \| < 2$.

[Hint: use a singular value system for A, $\{\sigma_i^2, u_i, v_i\}$, and the geometric sum formula $\Sigma_{k=0}^{n-1} z^k = (1 - z^n)/(1 - z)$.]

END OF PAPER