UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 9:00 am to 12:00 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Cartesian co-ordinates (x, y, z) *are used with* z *denoting the upward vertical. The corresponding velocity components are* (u, v, w)*. Unless stated otherwise,* g *is the gravitational acceleration.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider motion governed by the following linearized rotating shallow water equations on an f-plane (with a constant Coriolis parameter):

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y},$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0.$$

When the free surface is undisturbed, the depth of the fluid is

$$H = \begin{cases} H_0 - d, & y < 0 \\ H_0 + d, & y > 0 \end{cases}$$

where $d \ll H_0$. At $t = 0$ the fluid is at rest and the elevation of the free surface height is

$$\eta = \begin{cases} -\eta_0, & x < 0 \\ \eta_0, & x > 0 \end{cases}$$

where η_0 is constant and $\eta \ll H_0$.

Find the steady state solution for η valid for large values of $|y|$ (far from the step). From this solution, calculate the potential energy per unit length released during the adjustment from the initial conditions to the steady state. Obtain an expression for the volume flux towards and away from the step for large $|y|$.

Derive a partial differential equation describing the evolution of $\eta(x, y, t)$. Assuming that η and Hv are continuous at $y = 0$ and that disturbances to the steady state solution decay as $y \to \pm\infty$, find the dispersion relation for disturbances of the form

$$\eta'(x, y, t) = \mathrm{Re}\left(\widehat{\eta}(y)e^{i(kx-\omega t)}\right).$$

Obtain an explicit expression for ω in the quasi-geostrophic limit ($\omega \ll f_0$). Deduce the direction of the phase velocity of the waves in this limit.

From your expression in the previous part, consider additionally the limit of long waves where $k \ll 1/R_D$ where $R_D = \sqrt{gH_0}/f_0$. Find the group velocity and phase speed of these waves. As the long waves propagate away from the origin they leave behind a surface displacement of the form

$$\eta = \eta_0 \mathrm{sgn}(x) - A(x, t)e^{-|y|/R_D},$$

where sgn is the sign function. Find and solve a partial differential equation for $A(x, t)$. Calculate the volume flux associated with the flow along the step for $y < 0$ and $y > 0$ and compare the result with the volume flux calculated earlier for the current far from the step.

2 An otherwise motionless ocean in the domain $-\infty < z < 0$ is subject to localized heating from solar insolation such that the buoyancy evolves according to

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b = R(x, y, z),$$

where $b = -g\rho/\rho_0$ is the buoyancy and ρ_0 is a constant reference density. Assuming that the Rossby number associated with the subsequent motion is small, the Coriolis parameter is $f \simeq f_0 + \beta y$ with f_0 and β constant, and $b = N_0^2 z + b'$ where N_0^2 is constant, derive an equation describing the evolution of quasi-geostrophic potential vorticity. Clearly state any other approximations that you make.

If the buoyancy forcing in the previous equation is

$$R = \frac{-R_0 z}{H} \exp\left(\frac{-x^2 - y^2}{L^2} + \frac{z}{H}\right),$$

where R_0, H, and L are constant and $H > 0$, solve for steady small amplitude (linearised) circulation in geostrophic balance subject to the condition that the velocity vanishes for $x \to +\infty$. Sketch the streamlines corresponding to this circulation in x-y plane at $z = 0$. Discuss the circulation in the context of changes to the potential vorticity induced by the forcing.

Considering the nonlinear quasi-geostrophic equations, derive the following Ω equation for the vertical circulation

$$N_0^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f_0^2 \frac{\partial^2 w}{\partial z^2} = \text{RHS},$$

where the term(s) on the right hand side (RHS) should be determined. Now considering the response to weak forcing such that you can neglect any nonlinear terms appearing in RHS, sketch the circulation in a y-z plane at $x = 0$ *(you do not need to obtain an explicit solution to the Ω equation)*. What is the vertical circulation far from the forcing region?

3 Consider quasi-geostrophic flow on an f-plane, with constant Coriolis parameter f_0 and constant buoyancy frequency N. Explain, without detailed derivation of the quasi-geostrophic equations, why the leading-order approximation to the vertical velocity w is given by

$$w = -\frac{D_g}{Dt}\left\{\frac{f_0\psi_z}{N^2}\right\}$$

where ψ is the quasi-geostrophic stream function and D_g/Dt denotes the rate of change following the geostrophic flow. Show that the appropriate boundary condition on ψ at the rigid sloping boundary $z = \alpha y$, where α is comparable to the Rossby number, is

$$w = -\frac{D_g}{Dt}\left\{\frac{f_0\psi_z}{N^2}\right\} = \alpha\psi_x.$$

The surface $z = \alpha y$ forms a lower boundary for a semi-infinite domain in which there is a basic flow $(u, v, w) = (\Lambda z, 0, 0)$, where Λ is constant. Assuming that the boundary condition above may be applied at $z = 0$ (so the flow domain is $0 < z < \infty$), write down the equations governing small-amplitude disturbances to the basic flow in terms of the disturbance quasi-geostrophic stream function ψ'. Show that if the interior disturbance quasi-geostrophic potential vorticity $q' = \psi'_{xx} + \psi'_{yy} + (f_0^2/N^2)\psi'_{zz}$ is zero initially then it is zero for all time.

For disturbances with $q' = 0$ and of the form $\psi' = \mathrm{Re}(\hat{\psi}(z,t)e^{ikx})$ show that the time evolution of $\hat{\psi}_z(0,t)$ is described by an ordinary differential equation. Derive a dispersion relation for the phase speed c in terms of the x-wavenumber k when $\hat{\psi}(z,t) = \hat{\psi}^{(c)}(z)e^{-ikct}$. Give brief qualititative explanations of the form of the ordinary differential equation and of the propagation characteristics as captured by the dispersion relation, including how these characteristics depend on the vertical shear Λ and the boundary slope α.

Now consider the case where there is additionally a rigid upper boundary at $z = D$. Again consider disturbances of the form $\psi' = \mathrm{Re}(\hat{\psi}(z,t)e^{ikx})$, with $q' = 0$. Show that

$$\hat{\psi}(z,t) = -\frac{\cosh(\mu(z-D))}{\mu\sinh\mu D}\hat{\psi}_z(0,t) + \frac{\cosh(\mu z)}{\mu\sinh\mu D}\hat{\psi}_z(D,t),$$

where $\mu = Nk/f_0$.

Use this expression to deduce a coupled pair of ordinary differential equations for the quantities $\hat{\psi}_z(0,t)$ and $\hat{\psi}_z(D,t)$. Hence derive the dispersion relation for c in terms of k when $\hat{\psi}(z,t) = \hat{\psi}^{(c)}(z)e^{-ikct}$. You may find it helpful to define the non-dimensional quantities $\tilde{c} = c/\Lambda D$, $\tilde{\mu} = NkD/f_0$ and $\tilde{\alpha} = \alpha N^2/f_0\Lambda$. Deduce that the flow is stable when $\tilde{\alpha} > 1$ and unstable if $0 \leqslant \tilde{\alpha} < 1$. Can you say anything about $\tilde{\alpha} < 0$?

4 The Boussinesq, f-plane, small Rossby number form of the Eulerian-mean equations including momentum flux and density flux as forcing terms are as follows:

$$\overline{u}_t - f_0\overline{v}_a = -(\overline{u'v'})_y \tag{1}$$

$$f_0\overline{u} = -\frac{\overline{p}_y}{\rho_0} \tag{2}$$

$$\overline{\rho}g = -\overline{p}_z \tag{3}$$

$$\overline{v}_{ay} + \overline{w}_{az} = 0 \tag{4}$$

$$\overline{\rho}_t + \overline{w}_a\frac{d\rho_s}{dz} = -(\overline{\rho'v'})_y. \tag{5}$$

Overbars in these equations indicate averages in x, primes indicate disturbance quantities, i.e. departures from the x-average value, $(\overline{v}_a, \overline{w}_a)$ are the (y, z) components of the Eulerian-mean flow. The density is made up of three parts: ρ_0 which is constant, $\rho_s(z)$ which defines the background state stratification and ρ which is associated with disturbance from the resting background state, with p the corresponding pressure disturbance. $d\rho_s/dz$ is constant and N is the corresponding buoyancy frequency.

Starting from these equations, derive the transformed Eulerian-mean equations and explain the role of the Eliassen-Palm flux in these equations. Explain also the relation of the Eliassen-Palm flux to Rossby-wave propagation. (Detailed derivation of the Eliassen-Palm wave activity relation is not required.) State and explain a corresponding 'non-acceleration' theorem.

Now consider the effect on the mean flow of propagating and dissipating Rossby waves in the domain $0 < y < L$ (with rigid boundaries at $y = 0$ and $y = L$), $-\infty < z < \infty$, assuming that $\overline{u'v'} = 0$ and

$$\overline{\rho'v'} = -\frac{d\rho_s}{dz}\frac{G_0}{f_0}\sin(\pi y/L)\mathcal{F}(z),$$

where G_0 is a positive constant and

$$\mathcal{F} = \begin{cases} -1 & (z < -D), \\ (z - D)/2D & (-D < z < D), \\ 0 & (z > D), \end{cases}$$

i.e. the waves are excited by topography at a distant lower boundary, propagate upwards and dissipate in the region $-D < z < D$. Solve for the latitudinal component $\overline{v}_a^*$ of the transformed Eulerian-mean circulation and for the acceleration $\overline{u}_t$. Justify carefully any boundary conditions that you apply. [*You may assume that these and related quantities have the form of a function of z multiplied by either* $\sin(\pi y/L)$ *or* $\cos(\pi y/L)$, *according to the relevant boundary conditions.*]

Sketch the form of of $\overline{u}_t$ and $\overline{\rho}_t$ in the (y, z) plane showing clearly what sign they take in different regions. Similarly sketch the streamlines of the Eulerian-mean and transformed Eulerian-mean circulations showing the direction of each.

Calculate the quantities $\int_{-\infty}^{\infty} dz\, \overline{u}_t$ and $\int_{-D}^{D} dz\, \overline{u}_t$ and comment on their values, in particular on their variation with the quantity ND/f_0L.

END OF PAPER