UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 1:30 pm to 4:30 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A fluid of viscosity μ_1 is injected into a porous medium of porosity ϕ and uniform permeability k saturated in a fluid of viscosity $\mu_2 > \mu_1$. The initial interface between the two fluids is planar, and the influence of gravity on the system is negligible. If the fluid is injected into the medium with constant velocity U, show that the interface between the fluids is unstable to perturbations with growth rate

$$\sigma = \frac{\alpha U}{\phi} M$$

where $M = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$ is the mobility ratio, and α is the wavenumber of the instability.

The porous medium contains a spatially variable wetting angle which manifests as a macroscopic gradient in the apparent surface tension. Consider the role of a spatially variable surface tension acting between the fluids of the form

$$p_1 - p_2 = \gamma(1 + \beta x)(\kappa_0 + \nabla \cdot \hat{\mathbf{n}}),$$

where p_1, p_2 are the pressures in the two fluids at the interface, $\gamma(1 + \beta x)$ is the surface tension whose gradient in the direction of flow is β, κ_0 is an intrinsic curvature in the porous medium (at the pore-scale) and $\hat{\mathbf{n}}$ is the unit normal to the interface between the fluids. Find the dispersion relationship for the growth of perturbations along the interface as a function of M, the capillary number $Ca = \mu_2 U/\gamma$, and the gradient in surface tension β. What is the most unstable wavenumber? For what flow rates can the instability be suppressed entirely?

2

A layer of fluid of kinematic viscosity ν drains vertically into an infinitely deep porous medium of porosity ϕ and permeability k under the action of gravity g. At time t, the layer has thickness $h(t)$ above the porous medium and occupies a region of thickness $l(t)$ within the porous layer. Given that the volume of fluid is fixed and that $l(0) = 0$, $h(0) = h_0$, determine a differential equation governing the evolution of $l(t)$. Show that at early times $t \ll 1$

$$l(t) \sim \left(\frac{2kgh_0}{\phi\nu} t \right)^{1/2}.$$

A two-dimensional gravity current of total volume $V_0(t/\tau)^3$, where τ is a constant time scale, flows horizontally over a deep porous medium while draining vertically into the medium. From first principles and giving physical descriptions of your model equations, determine the equations governing the surface elevation $h(x,t)$ of the current and the depth $l(x,t)$ to which it has drained into the porous medium

$$\frac{\partial h}{\partial t} - \frac{1}{3}\frac{g}{\nu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x} \right) = -\left(\frac{gk}{\nu} \right)\left(1 + \frac{h}{l} \right) = -\phi \frac{\partial l}{\partial t}.$$

What constraints apply to these equations?

Show that your equations and constraints admit a family of similarity solutions parameterised by ϕ and a drainage parameter

$$K = k \left(\frac{g^3 \tau^3}{\nu^3 V_0} \right)^{2/5}.$$

(Note that you may find K raised to a power appearing in your dimensionless system of equations.) You should determine a suitable similarity variable, the functional dependence of the length of the current on time and the parameters of the system, up to a multiplicative constant, and the ordinary differential equations describing the solutions. You do not need to solve the equations.

3

Magma is injected into the shallow crust, forming an axisymmetric intrusion with an elastically deformed upper surface whose deformation can be modelled by that of a bending beam with stiffness B (hydrostatic pressure contributions should be ignored throughout this question). The magma initially enters the intrusion at a constant volumetric flux, Q, and flows with viscosity μ. At all times, the inflation of the intrusion is quasi-static, so that the interior pressure p is approximately uniform, and the rate of propagation is controlled by physical processes at the front. Show that, in the interior, the thickness of the intrusion is

$$h(r,t) = \frac{3V}{\pi R^2}\left(1 - \frac{r^2}{R^2}\right)^2,$$

where $V(t)$ is the volume of the intrusion and $R(t)$ is its radial extent.

At early times, a vapour tip extends from the fracture front to the fluid front, whose height is approximately given by

$$h_f \simeq \frac{p_T l_p^4}{24B},$$

where p_T is the pressure in the vapour tip, and l_p is peeling lengthscale at the fluid front. Using a scaling analysis of the lubrication flow near the fluid front to derive an expression for dR/dt and, by matching with the interior curvature, show that the radial extent of the intrusion is approximately given by

$$R \sim \left(\frac{B^3 Q^7}{p_T \mu^2} t^9\right)^{1/30}$$

At late times the fracture toughness at the front dominates propagation and imposes a curvature at the front, κ_f. In this limit, determine the radial extent of the intrusion as a function of the volume, $R(V)$.

Finally, at late times the magma feeding the intrusion comes from an over pressured reservoir whose pressure is $p_r = EV_r$, where V_r is the volume of the reservoir and E is the elasticity of the rock. If the flux of magma from the reservoir to the intrusion is $Q = \beta(p_r - p)$, and the volume of magma in the conduit is negligible, derive an expression for the volume of the intrusion, dV/dt, as a function of the initial reservoir volume, V_{r0}, the volume of the intrusion, V, and material properties of the intrusion and reservoir (β, E, B, κ_f). What is the final size of the intrusion?

4

Consider a full annual cycle of a single-category sea-ice model starting with an ice-free ocean at the height of summer, with the atmospheric temperature given by

$$T_A = T_m + \Delta T \cos(t/\tau),$$

where T_m is the freezing temperature of the ocean (the effect of salt is ignored in this question), ΔT is the amplitude of the atmospheric temperature variation, $\tau = 1$ year$/2\pi$ and t is time. You may assume that heat fluxes to the atmosphere and from the ocean are given by simple Newton's laws with heat-transfer coefficients λ_A and λ_O respectively and that the ocean temperature is fixed at $T_O > T_m$.

You should determine expressions for the surface temperature of the ocean or ice in contact with the atmosphere, the time t_1 at which ice begins to form, an implicit equation for the thickness of ice $h(t)$, the maximum ice thickness h_m and the time t_2 when that is reached. You may assume that the heat flux from the ocean is negligible while ice is growing.

Show, in particular, that

$$h_m \approx [1 + \sin(t_1/\tau)]^{1/2} \left(\frac{2\kappa\tau}{\mathcal{S}}\right)^{1/2} \quad \text{if} \quad \frac{\lambda_A}{k}\left(\frac{\kappa\tau}{\mathcal{S}}\right)^{1/2} \gg 1,$$

where $\kappa = k/\rho c_p$ is the thermal diffusivity of ice, k is the thermal conductivity of ice, ρ is the density of ice, c_p is the specific heat capacity of ice, and $\mathcal{S} = L/c_p\Delta T$, where L is the latent heat of fusion.

Determine the conditions on the parameters of the system that must be satisfied so that the heat flux is indeed ignorable while ice is growing.

Determine the thickness of ice at the end of the first year and hence show that perennial ice forms if

$$\frac{1}{\mathcal{S}} \frac{\lambda_A \kappa\tau}{k}\left(1 + \frac{\pi}{2}\frac{\lambda_O}{\lambda_A}\frac{T_O - T_m}{\Delta T}\right) < h_m.$$

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