UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 09:00 am to 12:00 pm

PAPER 331

HYDRODYNAMIC STABILITY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider an isolated interface (with surface tension γ) between two incompressible, irrotational inviscid fluids of different densities in a finite depth region between two horizontal impermeable boundaries, such that when the flow is at rest $\rho = \rho_1$ for $0 < z \leqslant L_1$ and $\rho = \rho_2 < \rho_1$ for $-L_2 \leqslant z < 0$. The displacement $\eta = B\exp[i(kx - \omega t)]$ (real part understood, with $k > 0$ real) of the interface away from its equilibrium position at $z = 0$ may be assumed to be sufficiently small and smooth that the problem may be linearized, and the problem can also be considered to be two-dimensional.

(i) Write down the appropriate conditions at the impermeable boundaries on the upper-layer velocity potential ϕ_1 as $z \to L_1$ and on the lower-layer velocity potential ϕ_2 as $z \to -L_2$.

(ii) Briefly explain why the appropriate boundary conditions to apply at $z = \eta$ are

$$\left.\frac{\partial \phi_{1,2}}{\partial z}\right|_{z=\eta} = \frac{D\eta}{Dt},$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \frac{1}{2}\rho_1 |\boldsymbol{\nabla}\phi_1|^2 + g\rho_1\eta = \rho_2 \frac{\partial \phi_2}{\partial t} + \frac{1}{2}\rho_2 |\boldsymbol{\nabla}\phi_2|^2 + g\rho_2\eta + P_2 - P_1,$$

where P_1 and P_2 are the pressures immediately above and below the interface.

(iii) You may assume that

$$P_2 - P_1 = -\gamma \frac{\frac{\partial^2 \eta}{\partial x^2}}{\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{3/2}}.$$

Using this expression, linearize the boundary conditions, and hence show that the frequency $\omega = kc = \omega_r + i\sigma$ (where σ is real, and c is the phase speed, in general complex) satisfies the dispersion relation

$$\omega^2 \left[\rho_1 \coth kL_1 + \rho_2 \coth kL_2\right] = -g(\rho_1 - \rho_2)k + \gamma k^3.$$

(b) In the limit when both $L_1 \to \infty$ and $L_2 \to \infty$, identify:

(i) the cutoff wavenumber k_c;

(ii) the maximum growth rate σ_m;

(iii) and the wavenumber k_m of perturbation associated with σ_m.

(c) Now consider the limit where $L_2 \to \infty$ and $k_c L_1 \ll 1$ (i.e. a thin bounded layer of dense fluid over an infinitely deep layer of less dense fluid).

(i) Identify the cutoff wavenumber for this flow.

(ii) Identify the maximum growth rate for this flow.

(iii) Identify the wavenumber of perturbation associated with the maximum growth rate.

(iv) Briefly compare these three quantities for this flow to the equivalent quantities for the flow considered in part (b) above.

2

Consider infinitesimal two-dimensional perturbations about a parallel shear flow in an inviscid unstratified fluid in a finite depth domain $z \in [-L, L]$ between impermeable boundaries:

$$\begin{aligned} \mathbf{u} &= \overline{U}(z)\hat{\mathbf{x}} + \mathbf{u}'(x,z,t); \; U_{\min} \leqslant \overline{U} \leqslant U_{\max}; \\ p &= \overline{p}(z) + p'(x,z,t), \\ \left[\mathbf{u}', p'\right] &= [\hat{\mathbf{u}}(z), \hat{p}(z)] \exp[ik(x-ct)]; \; \hat{\mathbf{u}}(z) = (\hat{u}, \hat{w}), \end{aligned}$$

where the wavenumber k is assumed real, and the phase speed $c = c_r + ic_i$ may in general be complex. The vertical velocity eigenfunction $\hat{w}$ satisfies the Rayleigh equation,

$$\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \frac{\hat{w}}{(\overline{U} - c)}\frac{d^2}{dz^2}\overline{U} = 0.$$

(a) Prove Howard's semicircle theorem, i.e. show that

$$\left[c_r - \frac{(U_{\max} + U_{\min})}{2}\right]^2 + [c_i - 0]^2 \leqslant \left[\frac{(U_{\max} - U_{\min})}{2}\right]^2.$$

(b) Now consider a piecewise-linear shear flow where $U_{\min} = -U_{\max}$:

$$\frac{\overline{U}}{U_{\max}} = \begin{cases} 1 & \zeta_L < \zeta \leqslant 1 \ (R1); \\ \frac{\zeta}{\zeta_L} & |\zeta| < \zeta_L \ (R2); \\ -1 & -1 \leqslant \zeta < -\zeta_L \ (R3); \end{cases},$$

where $\zeta = z/L$ and $\zeta_L = L_s/L < 1$.

(i) Applying appropriate boundary conditions at $\zeta = \pm 1$, write down the forms of the general solution $\hat{w}$ (involving four arbitrary constants) to the Rayleigh equation in the three regions $R1$, $R2$ and $R3$.

(ii) Write down the conditions which $\hat{w}$ must satisfy at $\zeta = \pm 1$ and $\zeta = \pm\zeta_L$.

(iii) Hence derive four equations for the four arbitrary constants.

(c) You are given that the simultaneous solution of those four equations leads to the dispersion relation

$$c^2 = 1 - \frac{\alpha\zeta_L(1+X^2)Y^2 + 2\alpha\zeta_L XY - XY^2}{\alpha^2\zeta_L^2\left[(1+X^2)Y + X(1+Y^2)\right]},$$

where $\alpha = kL$, $X = \tanh[\alpha\zeta_L]$ and $Y = \tanh[\alpha(1-\zeta_L)]$. By considering the limits $\alpha \to 0$ and $\alpha \to \infty$ or otherwise, derive a condition on ζ_L for the flow to be linearly unstable. (You may assume that c^2 is a monotonic function of α.)

3 Consider incompressible unstratified flow at finite Reynolds number with a steady background flow $U(y)$ (where y is chosen to be the wall-normal direction).

(a) You may assume three-dimensional normal mode forms for sufficiently small and smooth wall-normal velocity v and wall-normal vorticity η, i.e.

$$v(x,y,z,t) = \hat{v}(y)e^{i(\alpha x+\beta z-\omega t)}; \ \eta(x,y,z,t) = \hat{\eta}(y)e^{i(\alpha x+\beta z-\omega t)}.$$

For an appropriate choice of non-dimensionalization, show that $\hat{v}(y)$ and $\hat{\eta}(y)$ satisfy the Orr-Sommerfeld equation and the Squire equation respectively:

$$\left[(-i\omega + i\alpha U)(\mathcal{D}^2 - \kappa^2) - i\alpha\mathcal{D}^2 U - \frac{1}{Re}(\mathcal{D}^2-\kappa^2)^2\right]\hat{v} = 0; \ \mathcal{D} \equiv \frac{d}{dy};$$
$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(\mathcal{D}^2-\kappa^2)\right]\hat{\eta} = -i\beta\mathcal{D}U\hat{v}; \ \kappa^2 = \alpha^2 + \beta^2.$$

(b) Define *Orr-Sommerfeld modes* and *Squire modes*, and show that Squire modes are always damped.

(c) Consider two-dimensional flow with $\beta = 0$, and constant background flow $U(y) = U_0$. Show that $\hat{v}$ satisfies

$$\hat{v}(y) = a_1 e^{\alpha y} + a_2 e^{-\alpha y} + a_3 e^{\gamma y} + a_4 e^{-\gamma y},$$

where γ is to be determined, and a_1, a_2, a_3, a_4 are constants determined by appropriate boundary conditions on $\hat{v}$ and $\mathcal{D}\hat{v}$. The constants a_1, a_2, a_3, a_4 do not need to be determined.

(d) Now consider two-dimensional flow (once again with zero spanwise wavenumber) and (dimensional) constant shear flow $U^*(y^*) = S^* y^*$. You are given that the dimensional form of the normal mode is $v^*(x^*, y^*, t^*) = \hat{v}^*(y^*)e^{ik^*(x^*-c^*t^*)}$.

(i) Using k^* and S^* to non-dimensionalize, show that $\hat{v}$ satisfies

$$(y-c)(\mathcal{D}^2 - 1)\hat{v} = \frac{-i}{Re}(\mathcal{D}^2-1)^2\hat{v},$$

for an appropriate choice of Reynolds number.

(ii) Show that $\hat{\omega} \equiv (\mathcal{D}^2 - 1)\hat{v}$ is proportional to the spanwise vorticity.

(iii) Using the substitution $Z = e^{i\pi/6}(y - c - i\delta^3)/\delta$, where $\delta^3 = 1/Re$, show that $\hat{\omega}$ satisfies Airy's equation:

$$\frac{d^2}{dZ^2}\hat{\omega} - Z\hat{\omega} = 0.$$

(iv) You may assume that the general solution to Airy's equation is

$$\hat{\omega} = a_1 Ai(Z) + a_2 Bi(Z),$$

where Ai and Bi are Airy functions of the first and second kind, and a_1 and a_2 are arbitrary constants. Using variation of parameters or otherwise, find the general form for the solution $\hat{v}(y)$ with four arbitrary constants, determined by no-slip and no-flux boundary conditions imposed at $y = y_1$ and $y = y_2$. (You are not required to impose these conditions.)

4

(a) Consider the general $n-$dimensional linear system for perturbations from some base state $\mathbf{q}_b$:

$$\frac{d\mathbf{q}}{dt} = \mathsf{L}\mathbf{q}; \quad \mathbf{q} \in \mathbb{C}^n; \; \mathbf{q}(0) = \mathbf{q}_0 = \sum_{k=1}^{n} \phi_k \mathbf{v}_k,$$

where the $\mathbf{v}_k$ are the normalized eigenvectors of the square-invertible matrix L, ordered by the real part of the associated eigenvalues, i.e. $\mathrm{Re}(\lambda_1) \geqslant \ldots \geqslant \mathrm{Re}(\lambda_n)$.

(i) Define the *matrix exponential* $\mathsf{B} \equiv \exp[\mathsf{L}t]$ (You may assume that B is invertible.)

(ii) Define the *matrix norm* B.

(iii) If L is *normal*, show that the gain $G(t)$

$$G(t) \equiv \max_{\mathbf{q}(0)\neq 0} \frac{\|\mathbf{q}(t)\|^2}{\|\mathbf{q}(0)\|^2} = \exp[2Re(\lambda_1)t],$$

where $\|(\cdot)\|$ is the conventional Euclidean norm.

(iv) Define the *right singular vectors* $\mathbf{v}$, the *left singular vectors* $\mathbf{u}$, and the *singular values* σ of a matrix B.

(v) For general L, with $\mathsf{B} = \exp[\mathsf{L}t]$, show that $G(t) = \sigma_1^2(t)$, and identify the associated optimal initial condition $\mathbf{q}_0$.

(b) Now consider the linear system describing perturbations $\mathbf{x}$ from a base state $\mathbf{x}_b = \mathbf{0}$:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -i\omega_1 & a \\ c & -i[\omega_1 - b] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathsf{A}\mathbf{x},$$

where a, b, c, d and ω_1 are all real.

(i) Assume $\mathbf{x}(t) \propto \exp[-i\omega t]$. Derive a condition for $\mathbf{x}_b$ to be unstable.
Henceforth assume $a = 1$, $b = c = 10/Re$.

(ii) Show that $\omega \simeq \omega_1 \pm i\sqrt{10/Re}$ as $Re \to \infty$.

(iii) Calculate the eigenvalues of A and show that their properties as Re varies are consistent with the condition derived in (i).

(iv) Identify the value(s) of $Re = Re_N$ for which A is normal.

(v) Calculate the eigenvectors of A for $Re = Re_N$, and demonstrate that they are orthogonal.

(vi) Identify the initial relationship between $x_1(0)$ and $x_2(0)$ associated with the largest relative growth of the perturbation $\mathbf{x}$ when $Re = Re_N$.

END OF PAPER