UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2018 1:30 pm to 4:30 pm

PAPER 329

SLOW VISCOUS FLOW

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Explaining your choice of trial harmonic potential, find the velocity $\mathbf{u}(\mathbf{x})$, strain rate $\mathbf{e}(\mathbf{x})$ and vorticity $\boldsymbol{\omega}(\mathbf{x})$ in the Stokes flow due to a point force $\mathbf{F}\delta(\mathbf{x})$ acting at the origin of an unbounded viscous fluid.

A force-free couple-free rigid sphere of radius a is placed in an unbounded strain flow with uniform strain rate $\mathbf{E}$. Find the perturbation to the flow arising from the presence of the sphere.

Two rigid spheres of radius a are placed far apart in unbounded fluid, which is otherwise at rest. The first sphere is acted on by a force $\mathbf{F}$ and is couple free. The second sphere is force free and couple free. By considering the *leading-order* interactions, explain why the first sphere moves with velocity

$$\mathbf{U} = \mathbf{U}_0 - \frac{15a^4}{4R^6}(\mathbf{U}_0 \cdot \mathbf{R})\mathbf{R} + O(U_0 a^5/R^5), \qquad (1)$$

where $\mathbf{U}_0$ is the velocity that the first sphere would have if the second sphere were absent, and $\mathbf{R}$ is the vector distance between the centres of the spheres.

Assume that $\mathbf{U}_0 \cdot \mathbf{R} \neq 0$. Show that the dissipation is decreased by the presence of the second sphere. By considering $\mathbf{U} \cdot \mathbf{U}$, or otherwise, explain why (1) is consistent with the minimum dissipation theorem.

Find the leading-order change in the velocity of the first sphere if the second sphere is still force free, but now prevented from rotating by a suitable couple.

[*You may assume the Faxén formulae*

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \frac{a^2}{6}\nabla^2 \mathbf{u}_\infty\,, \quad \boldsymbol{\Omega} = \frac{\mathbf{G}}{8\pi\mu a^3} + \frac{1}{2}\boldsymbol{\omega}_\infty\,,$$

and you may quote the velocity for Stokes flow round a rotating sphere.]

2

State the reciprocal theorem for two Stokes flows in which the body forces are zero. Prove that the resistance matrix, giving the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted by a rigid body when moving with velocity $\mathbf{U}$ and angular velocity $\mathbf{\Omega}$ through surrounding viscous fluid, is both symmetric and positive definite.

The centreline of a long helical wire of thickness ϵb, with $\epsilon \ll 1$, is given in Cartesian coordinates by

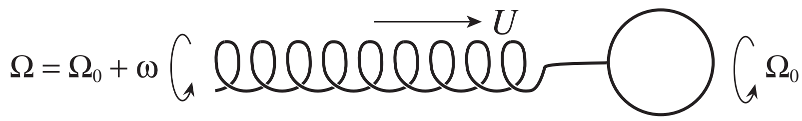
$$\mathbf{X}(\theta) = b(\cos\theta, \sin\theta, \theta\tan\phi), \qquad 0 \leqslant \theta \leqslant L\cos\phi/b, \qquad (*)$$

where ϕ is the pitch of the helix (i.e. the constant angle between $d\mathbf{X}/d\theta$ and the xy-plane) and $L\cos\phi \gg b$. Show that L is the total length of the wire.

By considering the magnitude and direction of the force exerted by a small line element, find the axial force F_z and couple G_z exerted by the helix in (i) pure translation in the z-direction with speed U and (ii) pure rotation about the z-axis with angular velocity Ω. [Assume that the non-axial components of $\mathbf{F}$ and $\mathbf{G}$ are zero.]

[*You may assume the slender-body formula* $\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \frac{1}{2}\mathbf{X}'\mathbf{X}')\cdot\mathbf{V}(\mathbf{X})$, *where* $C = 4\pi\mu/|\ln\epsilon|$, $\mathbf{X}' = d\mathbf{X}/ds$ *and* s *is the arc-length.*]

A micro-organism has the same density as the surrounding fluid. It consists of a long helical flagellum of the shape given by $(*)$ attached to a spherical head of radius a. A 'molecular motor' rotates the flagellum about its axis with angular velocity ω relative to the head. As a result, the micro-organism swims through the fluid with speed U, and both head and flagellum rotate, as shown below.



Assume that the hydrodynamic interactions between the helix and the sphere can be neglected, so that the resistance to motion of the micro-organism is the sum of the resistances of its separate parts. Find the flagellar rotation rate Ω and show that the swimming speed is given by

$$U = \frac{-BD_0\,\omega}{(A_0 + A)(D_0 + D) - B^2},$$

where A, B and D are the axial resistance coefficients of the helix, as calculated earlier, and $A_0 = 6\pi\mu a$ and $D_0 = 8\pi\mu a^3$ are the resistance coefficients of the sphere.

Find simplified expressions for U in terms of these coefficients for each of the regimes (i) $a \gg L$ and (ii) $a \ll (Lb^2)^{1/3}$. Explain physically why the micro-organism is a slow swimmer if the head is either too large or too small relative to the flagellum.

For the regime (iii) $(Lb^2)^{1/3} \ll a \ll L$, show that U is approximately independent of the size of the head, but has a maximum of $\omega b/2\sqrt{2}$ with respect to variations of the pitch angle ϕ. Comment briefly and physically on these results.

3

Surface tension should be neglected in all of these gravity-current problems.

(a) A rigid plane inclined at angle α to the horizontal is coated with a thin film of viscous fluid of thickness $h(x, y, t)$, where x is the down-slope coordinate and y the cross-slope coordinate. Use the equations of lubrication theory to derive the evolution equation for the film thickness.

(b) A fixed volume V of viscous fluid is released at the upwards-pointing apex of a cone whose sides slope downwards at an angle α to the horizontal. Assuming that the flow is axisymmetric, and taking x to be the distance down the sloping side, explain how your analysis in part (a) needs to be adapted to obtain the evolution equation for this situation.

After a long time the front of the current has travelled a distance $x_N(t) \gg (V \operatorname{cosec} \alpha)^{1/3}$. Find a similarity solution for the thickness $h(x, t)$ and determine $x_N(t)$ from mass conservation. [You do not need to analyse the detailed structure at the front.]

(c) In a two-dimensional flow, a thin film of viscous fluid of thickness $h(x, t)$ lies on top of a finite horizontal rigid boundary $-L \leqslant x \leqslant L$, and drains over the edges at $x = \pm L$. Use scaling arguments to show that $h \propto t^{-1/3}$ and to determine the expected form of the long-time similarity solution.

Assuming that the flow is described by lubrication theory subject to boundary conditions $h = 0$ at both edges, show that the similarity function $H(\eta)$ is given implicitly by an equation of the form

$$\eta(H) = \int_H^{H_0} F(s)\, ds\,,$$

where the function F should be determined. Hence determine $h_0 = h(0, t)$ in terms of the various parameters and the constant C given by the beta function

$$C = \int_0^1 \frac{du}{u^{1/5}(1-u)^{1/2}} = \mathrm{B}(\tfrac{4}{5}, \tfrac{1}{2})\,.$$

Show that $H \sim (1 - \eta)^k$ as $\eta \to 1$, where $0 < k < 1$ is to be found, and deduce that lubrication theory does not apply in a region close to $x = L$. Show further that the size of this region is given by $L - x \sim L(h_0/L)^{4/3}$, and comment on the appropriateness, or otherwise, of the boundary condition $h = 0$ used earlier.

END OF PAPER