UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 9:00 am to 11:00 am

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $X \subset \mathbf{R}^n$ be open. Define the class of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$. What does it mean for $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ to be a *phase function*?

If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ and Φ is a phase function explain how the oscillatory integral

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x,\theta)\,\mathrm{d}\theta,$$

defines a linear form on $\mathcal{D}(X)$. Show that $I_\Phi(a) \in \mathcal{D}'(X)$.

Write $D^\alpha \delta_{x_0}$ as an oscillatory integral, demonstrating explicitly that the two distributions are equal.

Can every element of $\mathcal{D}'(X)$ be written as an oscillatory integral? Give a proof or counterexample.

2

Define the space of test functions $\mathcal{D}(\mathbf{R})$ and the space of distributions $\mathcal{D}'(\mathbf{R})$, specifying the notion of convergence on each.

(i) Define the principal value $\mathrm{p.v.}(1/x)$ and show that it belongs to $\mathcal{D}'(\mathbf{R})$. Establish the identity

$$\mathrm{p.v.}(1/x) = \frac{\mathrm{d}}{\mathrm{d}x} \log|x| \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

(ii) For $\varphi \in \mathcal{D}(\mathbf{R})$ and $m = 2, 3, \ldots$ consider the linear forms

$$\langle \Lambda_m, \varphi \rangle = \lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} \left(x^{-m}\varphi(x) - \sum_{k=0}^{m-2} \frac{x^{k-m}}{k!} \varphi^{(k)}(0) \right) \mathrm{d}x.$$

Using this definition, show that each Λ_m defines an element of $\mathcal{D}'(\mathbf{R})$ of finite order. By establishing the identity

$$\left\langle \Lambda_m, \varphi' \right\rangle = m \left\langle \Lambda_{m+1}, \varphi \right\rangle,$$

or otherwise, show that

$$\Lambda_m = c_m \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^m \log|x| \quad \text{in } \mathcal{D}'(\mathbf{R}),$$

where the c_m are constants you should determine.

3

Let P be an Nth order polynomial in $\lambda = (\lambda_1, \ldots, \lambda_n)$. What does it mean to say that $P(D)$ is *elliptic*? Show that if $P(D)$ is elliptic then $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

Let $X \subset \mathbf{R}^n$ be open. Define the Sobolev space $H^s(\mathbf{R}^n)$ and local Sobolev space $H^s_{\mathrm{loc}}(X)$. Prove that if $u \in \mathcal{D}'(\mathbf{R}^n)$ has compact support then u belongs to a Sobolev space of sufficiently negative index.

Show that if $P(D)$ is an Nth order elliptic partial differential operator and $P(D)u \in H^s_{\mathrm{loc}}(X)$ then $u \in H^{s+N}_{\mathrm{loc}}(X)$. Elementary facts about Sobolev spaces can be used, provided they are clearly stated.

Give an example of a first order elliptic partial differential operator. Does there exist an elliptic differential operator in three variables (x_1, x_2, x_3) whose order is odd? Justify your answer.

END OF PAPER