UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2018 9:00 am to 11:00 am

PAPER 326

INVERSE PROBLEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Linear inverse problems and variational methods

This question deals with linear inverse problems and basic concepts of variational analysis. *You may use results from the lectures provided these are clearly stated.*

1. (a) State the definitions of *least squares solution* and *minimum norm solution*. When does a unique minimum norm solution exist? Give an example of an inverse problem where no minimum norm solution exists.

   (b) Let $\mathcal{U}$, $\mathcal{V}$ be Hilbert spaces, $K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and let $f \in \mathcal{V}$. Prove that the set

   $$\mathbb{L} = \{u \in \mathcal{U} \mid K^*Ku = K^*f\}$$

   is closed and convex. Moreover, show that it is non-empty if and only if $f \in \mathcal{R}(K) \oplus \mathcal{R}(K)^\perp$.

   (c) State the definition of the *Moore–Penrose inverse* and its connection to the minimum norm solution. When is the Moore–Penrose inverse continuous?

   (d) Show that, for $f \in \mathcal{D}(K^\dagger)$ and $u_0 \in \mathcal{U}$, there exists a unique least squares solution $u_0^\dagger$ of $Ku = f$ that has minimum distance to u_0, i.e.

   $$\|u_0^\dagger - u_0\|_\mathcal{U} = \min\{\|v - u_0\|_\mathcal{U} \mid v \text{ is a least squares solution to } Kv = f\},$$

   and show that it has the representation $u_0^\dagger = K^\dagger f + \mathcal{P}_{\mathcal{N}(K)} u_0$.

2. Consider a functional $E : \mathcal{U} \to (-\infty, +\infty]$ defined on a Banach space $\mathcal{U}$.

   (a) Define the concept of a *minimiser* of E. When do we call E *proper*, *coercive*, and *τ-sequentially lower semi-continuous (lsc)* for some topology τ on $\mathcal{U}$?

   (b) For each of the three properties (proper, coercive, and lsc) give an example of a functional that does not possess this property and does not have a minimiser.

   (c) Define the *epigraph* of E. Show that E is lsc if and only if the epigraph of E is sequentially closed in $(\mathcal{U}, \tau) \times \mathbb{R}$.

   (d) State the main existence theorem of minimisers, called the "Direct method" in the lecture, and prove it.

   (e) Let $\mathcal{U}$ be reflexive, $\mathcal{V}$ a Hilbert space, $K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be a bounded linear operator and $f \in \mathcal{V}$. Further, let $D : \mathcal{U} \to \mathbb{R}, D(u) := \|Ku - f\|_\mathcal{V}$. Use the direct method to prove that

   $$D \text{ is coercive} \quad \Rightarrow \quad f \in \mathcal{R}(K) \oplus \mathcal{R}(K)^\perp .$$

   Show that the reverse statement does not hold in general.

2 Regularisation

This question is concerned with regularisation of linear inverse problems. In particular, it deals with iterative regularisation. *You may use results from the lectures provided these are clearly stated.*

1. (a) State the general form of a *regularisation operator* R_α in terms of the singular value decomposition (SVD). Give two concrete examples of regularisation operators including one that does not require knowledge of the SVD.

   (b) Given a linear regularisation $\{R_\alpha\}_{\alpha>0}$, under what conditions does an a-priori parameter choice rule lead to a convergent regularisation?

   (c) State the *Landweber iteration* and state the corresponding regularisation operator in terms of the SVD.

2. Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces and let $K \in \mathcal{K}(\mathcal{U}, \mathcal{V})$ be a compact linear operator. Moreover, let $f \in \mathcal{D}(K^\dagger)$ and $f^\delta \in \mathcal{V}$ with $\|f - f^\delta\|_\mathcal{V} \leqslant \delta$ and $\delta > 0$. For $u_\delta^{(0)} := 0$ and $0 < \tau < \infty$ a step size parameter, *iterated Tikhonov regularisation* consists of computing for each $k \in \mathbb{N}$ the solution $u_\delta^{(k+1)}$ to

$$(I + \tau K^* K) u_\delta^{(k+1)} = u_\delta^{(k)} + \tau K^* f^\delta. \tag{1}$$

   (a) Derive the spectral representation of the operator $R_{\frac{1}{k}} f^\delta := u_\delta^{(k)}$. Prove that it is a linear regularisation.

   (b) Let $u^{(k)}$ be defined by (1) with f^δ replaced by f. For constants $C, \gamma > 0$, prove the estimates

$$\|u_\delta^{(k)} - u^{(k)}\|_\mathcal{U} \leqslant C\delta \quad \text{and} \quad \|K u_\delta^{(k)} - K u^{(k)}\|_\mathcal{V} \leqslant \gamma\delta.$$

   (c) Consider the noise-free scenario, i.e. $\delta = 0$. Assume $f = K u^\dagger$ and the existence of an element $v \in \mathcal{V}$ such that the source condition $u^\dagger = K^* v$ is satisfied. Show that

$$\|u^{(k)} - u^\dagger\|_\mathcal{U} = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$

   (d) Prove that with each iteration the residual is non-increasing, i.e. for all $k \in \mathbb{N}$,

$$\|K u_\delta^{(k+1)} - f^\delta\|_\mathcal{V} \leqslant \|K u_\delta^{(k)} - f^\delta\|_\mathcal{V}.$$

   (e) Prove that, as long as $\|K u_\delta^{(k+1)} - f^\delta\|_\mathcal{V} \geqslant \delta$, the iterates (1) satisfy

$$\|u_\delta^{(k+1)} - u^\dagger\|_\mathcal{U} \leqslant \|u_\delta^{(k)} - u^\dagger\|_\mathcal{U}.$$

   (f) State iteration (1) in variational form, i.e. as minimisation problem.

3 Variational regularisation

This question deals with variational regularisation, in particular with the concepts of value function calculus and infimal convolution. *You may use results from the lectures provided these are clearly stated.*

1. Let $\mathcal{U}$ be a Banach space, $\mathcal{V}$ a Hilbert space, and define the regularised solution

$$u_\alpha := \arg\min_{u \in \mathcal{U}} \Phi_{\alpha,f}(u)$$

with $\Phi_{\alpha,f}(u) := D(u) + \alpha J(u)$ consisting of the data term $D(u) = \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2$, where $K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $f \in \mathcal{V}$, and the regulariser $J : \mathcal{U} \to [0, \infty]$. We consider the same setting as in the lecture that guarantees the well-definedness of u_α for all $\alpha > 0$. In particular, we assume that $\Phi_{\alpha,f}$ is coercive and τ-lower semi-continuous (lsc) where the topology τ has the property that bounded sequences in $\mathcal{U}$ have τ-convergent subsequences. Recall that both D and J are lsc. For simplicity, assume $J(0) = 0$.

   (a) Show that for any sequence $\alpha_n \to \alpha > 0$ it follows that $u_{\alpha_n} \to u_\alpha$ in τ.

   (b) Let $\Psi(\alpha) := \Phi_{\alpha,f}(u_\alpha)$. Show that Ψ is non-decreasing, concave (i.e. $-\Psi$ is convex) and continuous.

   (c) Prove that $\alpha \mapsto J(u_\alpha)$ is non-increasing and that $\alpha \mapsto D(u_\alpha)$ is non-decreasing.

   (d) Show the estimate $J(u_\alpha) \leqslant \frac{1}{2\alpha}\|f\|_{\mathcal{V}}^2$, thus $\lim_{\alpha\to\infty} J(u_\alpha) = 0$. Give an example of a choice of J where this implies $\lim_{\alpha\to\infty} D(u_\alpha) = \frac{1}{2}\|f\|_{\mathcal{V}}^2$.

   (e) Let $f \in \mathcal{R}(K)$. Prove that $\lim_{\alpha\to 0} D(u_\alpha) = 0$ and $\lim_{\alpha\to 0} \alpha J(u_\alpha) = 0$.

2. The *infimal convolution* of two functionals $E, F : \mathcal{U} \to (-\infty, +\infty]$ is defined as

$$(E \,\square\, F)(u) = \inf_{v \in \mathcal{U}} E(v) + F(u - v)\,,$$

where we assume that $E \,\square\, F$ is proper and bounded from below.

   (a) Write down the definition of the *convex conjugate*.

   (b) Show that the infimal convolution $E \,\square\, F$ is convex if both E and F are convex.

   (c) Derive the convex conjugate of the infimal convolution $E \,\square\, F$ in terms of the convex conjugates of E and F.

   (d) Let $C \subset \mathcal{U}$ be a non-empty, closed, and convex subset of the Hilbert space $\mathcal{U}$ and define

$$d_C(u) := \inf_{v \in C} \frac{1}{2}\|u - v\|_{\mathcal{U}}^2\,.$$

   Show that d_C is convex and compute its convex conjugate. Give an explicit expression for $C = \{u \in \mathcal{U} \mid \|u\|_{\mathcal{U}} \leqslant 1\}$.

END OF PAPER