UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 9.00 am to 12:00 pm

PAPER 323

QUANTUM INFORMATION THEORY

Attempt ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $p := \{p_x\}_{x \in J}$ and $q := \{q_x\}_{x \in J}$ denote two probability distributions such that $p \ll q$. Their relative entropy is defined as

$$D(p||q) := \sum_{x \in J} p_x \log \left(\frac{p_x}{q_x} \right).$$

It can be shown that $D(p||q) \geqslant 0$. Using this fact, prove that the quantum relative entropy $D(\rho||\sigma)$ of two states ρ and σ, for which $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$, satisfies $D(\rho||\sigma) \geqslant 0$.

(ii) Let $\Lambda : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^{d'})$ be a linear map and let

$$J := (\Lambda \otimes \mathrm{id})|\Omega\rangle\langle\Omega|, \tag{1}$$

where

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle,$$

with $\{|i\rangle\}_{i=1}^{d}$ being an orthonormal basis in $\mathbb{C}^d$.

Prove that

(1) $\operatorname{Tr}(A\Lambda(B)) = d \operatorname{Tr}(J(A \otimes B^T))$.

[Hint: Recall that any bipartite pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A \simeq \mathbb{C}^d$ and $\mathcal{H}_B \simeq \mathbb{C}^k$, can be written in the form

$$(I_d \otimes R)|\Omega\rangle$$

for some $R \in \mathcal{B}(\mathbb{C}^d, \mathbb{C}^k)$.]

(2) Λ is completely positive if and only if $J \geqslant 0$;

(3) Λ is trace-preserving if and only if $\operatorname{Tr}_{\mathbb{C}^{d'}} J = \tau_d$, where τ_d denotes the completely mixed state in $\mathcal{D}(\mathbb{C}^d)$;

(4) Λ is unital if and only if $\operatorname{Tr}_{\mathbb{C}^d} J = I_{d'}/d$, where $I_{d'}$ denotes the identity operator in $\mathcal{B}(\mathbb{C}^{d'})$.

2 Consider a pair of quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$.

(i) Define the trace distance $D(\rho, \sigma)$ and the fidelity $F(\rho, \sigma)$.

(ii) Prove that $D(\rho, \sigma) = \max_{0 \leqslant \Lambda \leqslant I} \mathrm{Tr}(\Lambda(\rho - \sigma))$.

(iii) State and prove Uhlmann's theorem, clearly stating any other result that you use in the proof.

(iv) Let Λ denote a so-called *measure-and-prepare* quantum channel; its action on a state ρ is as follows. A measurement described by the POVM $\{E_m\}_{m=1}^k$ is performed on the state ρ and if the m^{th} outcome is obtained then the state $|m\rangle\langle m|$ is prepared. Write an expression for $\Lambda(\rho)$ and find a set of Kraus operators for Λ.

(v) Let $p_m = \mathrm{Tr}(E_m \rho)$ and $q_m = \mathrm{Tr}(E_m \sigma)$ denote the probabilities of getting the m^{th} outcome when a measurement, given by the POVM $\{E_m\}_{m=1}^k$ defined in (iv), is performed on the states ρ and σ respectively. Let $D(p, q) := \frac{1}{2}\sum_m |p_m - q_m|$ denote the classical trace distance between the probability distributions $p = \{p_m\}$ and $q = \{q_m\}$. Prove that $D(\rho, \sigma) \geqslant D(p, q)$.

[*Hint: Make use of (iv)*]

3 (i) The Holevo capacity (or Holevo information) of a memoryless quantum channel Λ is defined as follows:

$$\chi^*(\Lambda) := \max_{\{p_x, \rho_x\}} \big\{ S\big(\Lambda(\sum_x p_x \rho_x)\big) - \sum_x p_x S\big(\Lambda(\rho_x)\big) \big\}. \tag{1}$$

What is the operational significance of this quantity? Justify why the maximization can be restricted to ensembles of pure states, i.e., the states ρ_x can be chosen to be pure.

(ii) For a quantum channel $\Lambda : \mathcal{D}(\mathcal{H}_A) \to \mathcal{D}(\mathcal{H}_B)$, write an expression for $\chi^*(\Lambda)$ in terms of the mutual information.

(iii) Prove that the Holevo capacity is superadditive, i.e., for any pair of memoryless quantum channels Λ_1 and Λ_2,

$$\chi^*(\Lambda_1 \otimes \Lambda_2) \geqslant \chi^*(\Lambda_1) + \chi^*(\Lambda_2).$$

(iv) Let $\underline{x} = (x_1, \ldots, x_d)$ and $\underline{y} = (y_1, \ldots, y_d)$ be two d-dimensional real vectors. What is meant by the statement "$\underline{x}$ is majorized by $\underline{y}$ (denoted $\underline{x} \prec \underline{y}$)"? What is meant by $\rho \prec \sigma$ where ρ and σ are two states on a Hilbert space of dimension d?

(v) Let $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite pure state; let $d_A := \dim\mathcal{H}_A$ and $d_B := \dim\mathcal{H}_B$. Write its Schmidt decomposition and define its Schmidt rank. Prove that the Schmidt rank of $|\Psi_{AB}\rangle$ cannot be increased by LOCC, clearly stating any theorem that you use in your proof.

4 (i) Let ρ_{ABC} be a tripartite state. State the strong subadditivity (SSA) property of its von Neumann entropy $S(\rho_{ABC})$.

(ii) Let $I_c(\Lambda, \rho)$ denote the *coherent information* of a quantum channel $\Lambda : \mathcal{D}(\mathcal{H}_Q) \to \mathcal{D}(\mathcal{H}_B)$ when the input state is $\rho \in \mathcal{D}(\mathcal{H}_Q)$. Give its definition and prove that it satisfies the quantum data processing inequality, namely

$$S(\rho) \geqslant I_c(\Lambda_1, \rho) \geqslant I_c(\Lambda_2 \circ \Lambda_1, \rho),$$

where Λ_1 and Λ_2 are quantum channels, and $\Lambda_2 \circ \Lambda_1$ denotes the quantum channel obtained by their composition.

[*Remark: a clearly drawn and labelled figure would be helpful.*]

(iii) Let $\rho := \sum_{i,j=1}^{d} \rho_{ij} |i\rangle\langle j|$ denote a state in $\mathcal{D}(\mathbb{C}^d)$, and let $\tilde{\rho} \in \mathcal{D}(\mathbb{C}^d)$ be the following state:

$$\tilde{\rho} := \sum_{i=1}^{d} \rho_{ii} |i\rangle\langle i|.$$

Prove that the von Neumann entropy of these two states satisfy the following inequality: $S(\tilde{\rho}) \geqslant S(\rho)$.

5 (i) State the generalized measurement postulate in the case in which the initial state of the system to be measured is a mixed state. Under what condition does a generalized measurement reduce to a projective measurement?

(ii) Consider a measurement, described by a POVM $\{E_m\}_{m=1}^J$, on the state of a quantum system Q, which is initially in a state ρ, and let the measuring device D be initially in the pure state $|\varphi_D\rangle\langle\varphi_D|$. The Hilbert space $\mathcal{H}_D$ associated to the measuring device has dimension $\dim\mathcal{H}_D = J$. Such a measurement can be described by a quantum operation Λ which acts on the initial uncorrelated state of the composite system QD as follows:

$$\Lambda(\rho \otimes |\varphi_D\rangle\langle\varphi_D|) = \sum_m \sqrt{E_m}\rho\sqrt{E_m} \otimes |m_D\rangle\langle m_D|,$$

where $\{|m_D\rangle\}_{m=1}^J$ is an orthonormal basis in $\mathcal{H}_D$. Find a set of Kraus operators of Λ.

(iii) Consider the following two states

$$\rho_{XQD} := \sum_{x=1}^J p_x |x_X\rangle\langle x_X| \otimes \rho_x \otimes |\varphi_D\rangle\langle\varphi_D|,$$

and

$$\sigma_{XQD} := \sum_{x,m=1}^J p_x |x_X\rangle\langle x_X| \otimes \sqrt{E_m}\rho_x\sqrt{E_m} \otimes |m_D\rangle\langle m_D|,$$

where $\{|x_X\rangle\}_{x=1}^J$ denotes an orthonormal basis in a Hilbert space $\mathcal{H}_X$ of dimension J and $\rho_x \in \mathcal{D}(\mathcal{H}_Q)$ for each $x = 1, 2, \ldots, J$. Prove that

$$I(X : D)_\sigma \leqslant I(X : Q)_\rho,$$

carefully justifying your steps.

(iv) Let $\rho = \sum_{i=1}^n p_i\rho_i$, where the states ρ_i have mutually orthogonal supports. Prove that

$$S(\sum_{i=1}^n p_i\rho_i) = H(p) + \sum_{i=1}^n p_i S(\rho_i),$$

where $H(p)$ is the Shannon entropy of the probability distribution $\{p_i\}_{i=1}^n$.

(v) Prove that $I(X : Q)_\rho$ is equal to the Holevo χ-quantity of the ensemble $\{p_x, \rho_x\}_{x=1}^J$.

END OF PAPER