

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than **TWO** questions.

There are **THREE** questions in total.

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Accretion on to compact objects

(a) Consider a disk around a black hole of mass M accreting at a constant rate \dot{M} . Recalling that the Schwarzschild radius is $2GM/c^2$ (where c is the speed of light), show that the energy released per unit time via accretion is $\sim \dot{M}c^2$.

(b) Next consider a magnetised neutron star of mass M exhibiting a dipole field. The star is encircled by a disk aligned with the dipole so that the magnetic field strength at the disk midplane is $B \sim \mu/r^3$, where μ is the dipole moment. The disk accretes at a rate $\dot{M} = 2\pi r \Sigma v$ (where Σ is the disk's surface density and v is the radial speed of the accretion flow) and its angular thickness $\epsilon = H/r$ is constant, where H is the disk scale height.

The star's magnetic field truncates the disk at a radius r_m . Suppose that the disk is disrupted when magnetic pressure $B^2/(8\pi)$ is greater than the 'ram pressure' $\frac{1}{2}\rho v^2$ of an accretion flow in radial free fall from infinity (where ρ is density). Hence show that

$$r_m \sim \left(\frac{\epsilon^2 \mu^4}{GM\dot{M}^2} \right)^{1/7}.$$

(c) The magnetic dipole rotates at a rate Ω_m . Near r_m the disk plasma will be accelerated to this rate and possibly flung from the system in what is termed 'propeller flow'. Give a condition on r_m for this to occur.

(d) Briefly comment on the long-term evolution of the neutron star due to its magnetic connection to the disk.

(e) The evolution of the disk is governed by the following equations

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial}{\partial r} \left[\left(\frac{dh}{dr} \right)^{-1} \frac{\partial \mathcal{G}}{\partial r} \right], \quad \mathcal{G} = -2\pi \bar{\nu} \Sigma r^3 \frac{d\Omega}{dr},$$

where $h(r)$ is the specific angular momentum, \mathcal{G} the viscous torque, and $\bar{\nu}$ the turbulent kinematic viscosity (a constant). The disk receives mass at its outer boundary r_{out} at a rate \dot{M} , and receives a nonzero magnetic torque T_m at its inner radius $r = r_{\text{in}}$ (not necessarily r_m).

Assume the disk is in Keplerian rotation and in steady state. Find an expression for Σ in terms of $\bar{\nu}$ and the dimensionless parameter $\lambda = T_m/[h(r_{\text{in}})\dot{M}]$. Plot Σ as a function of r for $\lambda \gg 1$ and $\lambda \ll 1$. What regimes do these cases correspond to?

Suppose $r_{\text{in}} = r_m$ and the magnetic torque T_m is approximately $r^2 B^2 H/(4\pi)$ evaluated at r_m , i.e. the azimuthally and vertically averaged Maxwell stress. Using part (b), find an estimate of the value of λ .

2 Vertical structure of a slowly cooling disk

The governing equations of a thin astrophysical disk composed of ideal gas may be written as

$$D_t \rho = -\rho \nabla \cdot \mathbf{u}, \quad D_t \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi,$$

$$\frac{1}{\gamma - 1} \left(D_t P - \frac{\gamma P}{\rho} D_t \rho \right) = \mathcal{H} - \mathcal{C}, \quad P = \frac{k}{\mu m_p} \rho T,$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the total derivative, and ρ , \mathbf{u} , P , and T are density, velocity, pressure, and temperature respectively, while γ , k , μ , and m_p are adiabatic index, Boltzmann's constant, molecular weight, and the mass of a proton. The heating and cooling rates are $\mathcal{H} = (9/4)\alpha\Omega P$ and $\mathcal{C} = AT^{1+\beta}$, where Ω is the disk's orbital frequency and α , A and β are positive constants. Finally, the gravitational potential of the central star is $\Phi = -MG/r$, where M is the star's mass, and r is spherical radius.

(a) Suppose the disk is thin. Give the approximate expression for Φ usually employed when describing thin disks.

(b) At a fixed cylindrical radius, write down the equations controlling the equilibrium vertical structure of the disk. Solve these equations, given that $T = T_0$ (a constant) at $z = 0$. Write your solutions for T , ρ , and P in terms of disk semi-thickness H defined so that there is vacuum for $|z| > H$.

(c) Suppose the equilibrium is slightly perturbed. Give an order of magnitude treatment showing that the timescale upon which vertical equilibrium is re-established is $\sim \Omega^{-1}$.

(d) Consider now a non-turbulent disk that is cooling on a very long timescale $\gg \Omega^{-1}$. Using part (c), argue that the following equations adequately describe the slow evolution of the disk's vertical structure:

$$\partial_z P = -\Omega^2 z \rho, \quad \frac{1}{\gamma - 1} \left(D_t P - \frac{\gamma P}{\rho} D_t \rho \right) = -AT^{1+\beta},$$

with $P = k\rho T/(\mu m_p)$.

(e) Change independent variables from (z, t) to (ξ, t) , where the new similarity variable ξ is defined through $z = \xi\eta(t)$, with η yet to be determined but satisfying $\eta = 1$ at $t = 0$. Next assume the solution has the following form:

$$\rho = \tilde{\rho}(\xi)/\eta(t), \quad T = \tilde{T}(\xi)\eta(t)^2, \quad P = \tilde{P}(\xi)\eta(t).$$

Show that

$$\eta = (1 + Ct)^{-1/(1+2\beta)},$$

where C is a positive constant. Write down a set of equations in ξ for $\tilde{\rho}$, \tilde{T} , and \tilde{P} .

Using part (b), or otherwise, solve these equations given that at $t = 0$ the midplane temperature is T_0 . Find an expression for $T(z, t)$ in terms of a time-dependent semi-thickness $H(t)$, the form of which you should give. Briefly describe in words how the disk evolves.

3 Oscillatory convection in protoplanetary disks

Consider an accretion disk with a radially varying thermal structure. A local patch of the disk may be represented in the shearing sheet by the governing equations

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\rho_0} \nabla P - 2\Omega \mathbf{e}_z \times \mathbf{u} + 3\Omega^2 x \mathbf{e}_x - N^2 \theta \mathbf{e}_x, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= u_x + \xi \nabla^2 \theta, \quad \nabla \cdot \mathbf{u} = 0,\end{aligned}$$

where \mathbf{u} , P , and ρ_0 are the velocity, pressure, and density, respectively. In addition, θ is the ‘potential temperature’ perturbation, and ξ is thermal diffusivity. Finally, Ω is the orbital frequency of the shearing sheet and N is the radial buoyancy frequency of the gas. All of ρ_0 , Ω , N , and ξ are constants.

(a) Derive the vorticity equation:

$$\partial_t \boldsymbol{\omega} + \nabla \cdot \mathbf{T} = -N^2 \nabla \theta \times \mathbf{e}_x,$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u} + 2\Omega \mathbf{e}_z$, and the components of the tensor \mathbf{T} are given by $T_{ij} = u_i \omega_j - \omega_i u_j$. What can be said about the x -component of vorticity?

[You may need the identities:

$$\begin{aligned}\frac{1}{2} \nabla \mathbf{A}^2 &= \mathbf{A} \cdot \nabla \mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{A}, \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}.\end{aligned}$$

(b) Demonstrate that a steady solution to the governing equations is $\mathbf{u} = -(3/2)\Omega x \mathbf{e}_y$, $\theta = 0$, and $P = \text{a constant}$.

(c) Perturb this steady state with disturbances \mathbf{u}' , P' , and θ' proportional to $\exp(st + ikz)$, where s is a (possibly complex) growth rate and k a (real) vertical wavenumber.

Write down the linearised equations governing the perturbations. Show that $u'_z = P' = 0$. Hence derive the dispersion relation:

$$s^3 + \beta s^2 + (N^2 + \Omega^2)s + \beta \Omega^2 = 0,$$

where $\beta = \xi k^2 > 0$.

(d) Suppose that $n^2 = N^2/\Omega^2$ and $|n^2| \ll 1$. Consider the expansion, $s = s_0 + s_1 n^2 + \dots$

Show that the dispersion relation supports two epicycles with $s_0 = \pm i\Omega$ plus a third energy mode that decays at a rate you must find.

Discard the third mode and find the next order correction s_1 for the two epicyclic modes. Show that their instability criterion is $N^2 < 0$ and that their maximum growth rate to leading order is $-(1/4)N^2/\Omega$.

END OF PAPER