UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 9:00 am to 12:00 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Derive the virial theorem in the form

$$\tfrac{1}{2}\,\ddot{I} = 2\mathcal{T} + 3\int_{\text{star}} P\,\mathrm{d}V - 3P_{\mathcal{S}}\,V + \Omega,$$

where I is a moment of inertia, $\mathcal{T}$ is the internal kinetic energy, P is the pressure, Ω is the gravitational energy of all mass elements $\mathrm{d}m$ of the star and integration is over the star volume V, and the subscript $_{\mathcal{S}}$ refers to the stellar surface. State all necessary assumptions in your derivation and carefully define I, $\mathcal{T}$ and Ω.

You may start from

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{1}{\varrho}\,\nabla\cdot\mathbb{P} + \mathbf{F},$$

where $\mathbb{P}$ is the general (symmetric) stress tensor, to be defined, $\mathbf{F} = -\nabla\Phi$ for an isolated star, for which the gravitational potential Φ is given by $\Phi = -\displaystyle\int_{\text{star}} \frac{G\mathrm{d}m'}{|\mathbf{r}-\mathbf{r}'|}$.

(ii) Consider an ideal gas with ratio of specific heats at constant pressure and volume $\gamma = \dfrac{c_{\mathrm{p}}}{c_{\mathrm{v}}} = \text{const}$ and find the pressure integral in the derived virial theorem. For a steady state star composed of fully ionised gas obtain a relation between Ω and U, where U is the total internal energy.

(iii) Show that a polytrope of index n has

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R}.$$

and find expressions for U and Ω in terms of $\dfrac{G\,M^2}{R}$ and γ.

2

(i) Consider a low-mass star with a thin radiative envelope of material that behaves as an ideal gas and a convective region below. In the envelope $\nabla = \dfrac{\mathrm{d}\,\log T}{\mathrm{d}\,\log P} = \nabla_{\mathrm{rad}}$ and the opacity is given by $\kappa = \kappa_0 \varrho^n \, T^{-s}$. Find the pressure P as a function of temperature T in the envelope with $P = P_{\mathrm{phot}}$ at the photosphere, where κ_0 is a constant, ϱ the density and T is the temperature. Determine the variation of ∇ in the envelope in the form $\nabla = \dfrac{1}{C}\dfrac{P^{\mathrm{a}}}{T^{\mathrm{b}}}$, with constant C and exponents a and b to be determined.

(ii) The photospheric pressure $P_{\mathrm{phot}} = \frac{2GM}{3\kappa_{\mathrm{phot}} R^2}$, where M is the total mass and R the photospheric radius of the star. Show that $\nabla_{\mathrm{phot}} = \frac{1}{8}$.

(iii) Find an expression for $\nabla(r)$ as a function of $\dfrac{T_{\mathrm{eff}}}{T(r)}$ in the envelope when H^- ion opacity is dominant and given by $\kappa = \kappa_0 \, \varrho^{0.5} \, T^9$ and T_{eff} is the photospheric temperature.

(iv) Find $\dfrac{T_{\mathrm{eff}}}{T}$ when convection sets in and comment on the extent of the radiative envelope below the photosphere and the role of H^- opacity.

$\left[\textit{You may use } \left(\frac{8}{5}\right)^{2/9} \approx 1.11\right]$

3

(i) Consider a grey radiative atmosphere of a cool star in hydrostatic equilibrium. The stellar material is in local thermodynamic equilibrium. By considering the change in momentum of photons interacting with matter as they escape from the atmosphere show that the radiation pressure falls of according to

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\kappa\,\varrho\,L}{4\pi r^2 c},$$

where κ is the opacity, ϱ the density, L the luminosity, and P_{rad} is radiation pressure.

(ii) The optical depth is $\tau = -\int_0^\tau \kappa\varrho \mathrm{d}r$. Use Eddington's first approximation to find the radiation pressure when $\tau = 0$ and thence deduce that:

$$T^4(\tau) = \frac{1}{2}T_{\mathrm{eff}}^4\big(1 + \frac{3}{2}\tau\big),$$

where T is the temperature and T_{eff} is the photospheric temperature.

(iii) Show that $\dfrac{\mathrm{d}\log T}{\mathrm{d}\tau} = \dfrac{3}{8+12\tau}$.

(iv) For electron scattering $\kappa = \mathrm{const}$, show that $\dfrac{\mathrm{d}\log P}{\mathrm{d}\tau} = \dfrac{1}{\tau}$.

(v) Define the adiabatic exponent Γ_2. Show that if $\Gamma_2 < \frac{4}{3}$ convection must set in at large optical depth.

4

Discuss the evolution of a $5\,M_\odot$ star. Indicate important points in the evolution on a Hertzsprung–Russell diagram and give estimates of the age of the star at these points. Describe changes in the composition profiles of hydrogen and helium during the evolution. Pay particular attention to the Schönberg–Chandrasekhar limit, the evolution across the Hertzsprung gap, dredge-up events, blue loops, the Cepheid instability strip and thermal pulses.

END OF PAPER