UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 9:00 am to 12:00 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{4}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{5}$$

Conservation laws for momentum

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \hat{\Pi} = 0, \quad \hat{\Pi}_{ij} = \rho u_i u_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi}, \tag{6}$$

and energy

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{u^2}{2} + e \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + h \right) + c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0. \tag{7}$$

You may assume that for any scalar function f

$$\nabla f = \frac{\partial f}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z \quad \textit{(cylindrical coordinates)} \tag{8}$$

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi \quad \textit{(spherical coordinates).} \tag{9}$$

You may assume that for any vector $\mathbf{C}$

$$(\nabla \times \mathbf{C}) \times \mathbf{C} = (\mathbf{C} \cdot \nabla)\mathbf{C} - \frac{1}{2} \nabla \left(|\mathbf{C}|^2 \right), \tag{10}$$

$$\nabla \cdot \mathbf{C} = \frac{1}{R} \frac{\partial (R C_R)}{\partial R} + \frac{1}{R} \frac{\partial C_\phi}{\partial \phi} + \frac{\partial C_z}{\partial z} \quad \textit{(cylindrical coordinates)} \tag{11}$$

$$\nabla \cdot \mathbf{C} = \frac{1}{r^2} \frac{\partial (r^2 C_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (C_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial C_\phi}{\partial \phi}$$
$$\textit{(spherical coordinates).} \tag{12}$$

For any vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\nabla \cdot \mathbf{D}) + (\mathbf{D} \cdot \nabla)\mathbf{C} - \mathbf{D}(\nabla \cdot \mathbf{C}) - (\mathbf{C} \cdot \nabla)\mathbf{D}, \tag{13}$$

$$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot (\nabla \times \mathbf{C}) - \mathbf{C} \cdot (\nabla \times \mathbf{D}). \tag{14}$$

You may refer to these formulae in your solutions, but, please, make sure to provide sufficient details when using them.

1 (a) Derive the equation describing the time evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ in an ideal, isentropic, unmagnetized fluid, neglecting gravity:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \, .$$

(b) Derive the conservation law for the kinetic helicity $\mathbf{u} \cdot \boldsymbol{\omega}$ in the form

$$\frac{\partial}{\partial t} (\mathbf{u} \cdot \boldsymbol{\omega}) + \nabla \cdot \mathbf{F}_{H_k} = 0,$$

where $\mathbf{F}_{H_k}$ is the flux of kinetic helicity, and show an explicit expression for $\mathbf{F}_{H_k}$.

(c) Find the conditions under which kinetic helicity $\mathbf{u} \cdot \boldsymbol{\omega}$ is conserved in the Lagrangian sense, i.e.

$$\frac{d}{dt} (\mathbf{u} \cdot \boldsymbol{\omega}) = 0.$$

2 (a) Consider a 1-dimensional flow in a polytropic gas with adiabatic index γ. A stationary shock at $z = 0$ separates region 1 ($z > 0$, where $\rho = \rho_1$, $p = p_1$ and $u = u_1$) from region 2 ($z < 0$, where $\rho = \rho_2$, $p = p_2$ and $u = u_2$). Derive the jump conditions across the shock (Rankine-Hugoniot relations) in the form

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2, \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2, \\ u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma p_1}{\gamma - 1} \right) &= u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma p_2}{\gamma - 1} \right). \end{aligned}$$

Provide physical motivation for these relations.

Adopting the limit of a strong shock ($M_1 \gg 1$, $M_1^2 = u_1^2/(\gamma p_1/\rho_1)$) in these relations, derive expressions for ρ_2/ρ_1, u_2/u_1, and $p_2/(\rho_1 u_1^2)$.

(b) Massive young stellar clusters contain many massive stars in a small volume of space, that explode as supernovae over an interval of time. This results in a *continuous* injection of energy in a small volume (rather than *instantaneous* as in the case of a single supernova explosion). Such multiple explosions result in the so-called *superbubble* phenomenon — formation of a bubble of hot expanding gas, separated from the surrounding medium by a fast, strong shock.

Consider a model of a (spherically-symmetric) superbubble evolution in which the injection of energy can be described as

$$E(t) = \dot{E}t, \quad \dot{E} = const, \tag{1}$$

and there are no radiation losses. A strong shock separating expanding gas from the ambient medium of density ρ_1 and negligible pressure is located at $R(t)$.

Adiabatic expansion of the superbubble can be described by a similarity solution behind the shock in the form

$$\rho(r,t) = \rho_1 f(\eta), \quad p(r,t) = \rho_1 \dot{R}^2 g(\eta), \quad \text{and} \quad u(r,t) = \dot{R} h(\eta), \tag{2}$$

where the (dimensionless) functions f, g, and h depend only on a single similarity variable $\eta = r/R(t)$, with

$$R(t) = \alpha \dot{E}^a \rho_1^b t^c. \tag{3}$$

Determine the values of the constants a, b, and c.

(c) Derive the self-similar version of the continuity equation for the superbubble problem, i.e. show, using results (2)-(3), that the continuity equation reduces to an equation containing only the functions f, h, and the similarity variable η.

Formulate boundary conditions for f, g, h.

(d) Derive the expression allowing one to fix the value of the constant factor α in equation (3). What physical conservation law does it use?

3 (a) An axisymmetric distribution of dark matter gives rise to the potential in the form (in cylindrical coordinates)

$$\Phi(R,z) = \Phi_0 \left(\frac{R_0}{\sqrt{R^2 + \lambda^2 z^2}} \right)^{\beta}, \tag{1}$$

where $\Phi_0 < 0$, R_0, λ and $0 < \beta < 1$ are constants. Compute the distribution of dark matter density $\rho(R,z)$ that gives rise to such a potential.

A thin, cold, axisymmetric, non-self-gravitating gaseous disc orbits at the midplane of this potential (at $z = 0$). Neglecting any possible magnetic and thermal stresses, compute the angular frequency $\Omega(R)$, at which the disc rotates at a distance R.

(b) The disc from part (a) is threaded by the global axisymmetric magnetic field characterized by the magnetic flux function $\Psi(R,z)$. Write down how the components of the poloidal field $\mathbf{B}_p = (B_R, B_z)$ are related to Ψ.

It is known that in some interval of radii the behavior of the flux function at the disk surface is given by

$$\Psi(R,0) = \Psi_1 + \Psi_0 \left(\frac{R}{R_0} \right)^{\delta}, \tag{2}$$

where Ψ_1, Ψ_0, $\delta > 0$ are constants. Assume also that at $z = 0$ the poloidal magnetic field lines make an angle α (independent of R in this region) with respect to the vertical axis ($B_R > 0$).

Find how the B_z component of the field varies with height z near the $z = 0$ plane by computing $\partial B_z / \partial z$ at $z = 0$.

(c) The field threading the disc is strong enough to potentially launch a cold wind from its surface. Show that there is a minimum value of the field line inclination angle α near the disk midplane, for which this becomes possible, given the potential $\Phi(R,z)$ of this problem. Demonstrate that α is a function of λ and β and find the explicit dependence.

4 (a) Consider an axisymmetric MHD configuration, which is in differential rotation with a prescribed angular speed $\Omega(R,z)$, so that $\mathbf{u} = R\,\Omega(R,z)\mathbf{e}_\phi$. Show that the magnetic field's evolution is described by the following equation:

$$\frac{\partial \mathbf{B}}{\partial t} = R\left(\mathbf{B}_p \cdot \nabla\Omega\right)\mathbf{e}_\phi, \tag{1}$$

where $\mathbf{B}_p$ is the poloidal component of the magnetic field.

(b) Consider an axisymmetric disc orbiting in a point mass potential of a central star (in $z=0$ plane). Imagine, that at time $t=0$ an initial configuration of a weak field inside the disk is somehow established in the form

$$\mathbf{B}(R, t=0) = B_0 \frac{R_0}{R}\mathbf{e}_R, \tag{2}$$

(i.e. purely radial field) where B_0 and R_0 are constants. Determine the subsequent time evolution of the field driven by the rotation of the disc by finding the solutions for the poloidal and toroidal field components as a function of R and t. Here and below neglect the back-reaction of the magnetic field on the disc rotation, as well as the self-gravity of the disc.

Show that at every radius R, the toroidal field component B_ϕ becomes comparable in amplitude to the magnitude of the poloidal field $|\mathbf{B}_p|$ after a time equal to a fixed fraction of the local orbital time.

(c) Suppose further that the radial profile of thermal gas pressure in the disc is

$$p(R) = p_0\left(\frac{R_0}{R}\right)^2, \tag{3}$$

where $p_0 \gg B_0^2/(8\pi)$ is a constant.

Show that, as a result of field evolution, at late times there is always a radius $R_{\rm eq}(t)$ at which magnetic and thermal pressures are equal, and magnetic pressure dominates for $R < R_{\rm eq}$.

Show that $R_{\rm eq}(t) \propto t^\zeta$ and determine the value of the exponent ζ.

END OF PAPER