UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 9:00 am to 12:00 pm

PAPER 312

ADVANCED COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the Newtonian-gauge metric perturbation (negative gravitational potential) φ in the presence of local-type primordial non-Gaussianity

$$\varphi(\vec{x}) = \varphi_{\rm G}(\vec{x}) + f_{\rm NL}\left(\varphi_{\rm G}^2(\vec{x}) - \left\langle\varphi_{\rm G}^2(\vec{x})\right\rangle\right) ,$$

where the total metric perturbation gets contributions from the Gaussian potential $\varphi_{\rm G}$ and self-coupling of the Gaussian potential with coupling strength $f_{\rm NL}$.

(i) Splitting the perturbations into a long- and a short-wavelength part $\varphi = \varphi_{\rm s} + \varphi_{\rm l}$, show that at leading order in short and long modes (ignoring squares of short modes)

$$\varphi_{\rm s} = \varphi_{\rm G,s}(1 + 2f_{\rm NL}\varphi_{\rm G,l}) .$$

Squaring the short part of the *density* fluctuations, show that the short-scale variance of the density gets modulated as $\sigma_{\rm s}^2 = \left\langle\delta_{\rm s}^2\right\rangle_{\rm s} = \sigma_{\rm G,s}^2(1 + 4f_{\rm NL}\varphi_{\rm G,l})$. Here $\langle\cdot\rangle_{\rm s}$ denotes an average over realisations of the short modes at fixed long mode. For this purpose, use that the metric perturbation is related to the matter overdensity δ in Fourier-space by the Poisson equation $\alpha(k)\varphi = \delta$, where $\alpha(k) = 2k^2/(3\mathcal{H}^2\Omega_{\rm m})$. For the purposes of this derivation you may consider $\varphi_{\rm G,l}$ as a *constant background* over the support of the short modes.

(ii) The number density of haloes $n(\nu)$ is a function of the peak height $\nu = \delta_{\rm c}/\sigma_{\rm G,s}$. In the presence of long-wavelength density fluctuations $\delta_{\rm l}$, the collapse threshold gets lowered $\delta_{\rm c} \to \delta_{\rm c} - \delta_{\rm l}$ and the modulation of the short-scale variance in the presence of primordial non-Gaussianity leads to an additional dependence on the long-wavelength gravitational potential $\sigma_{\rm G,s} \to \sigma_{\rm G,s}(1 + 2f_{\rm NL}\varphi_{\rm G,l})$, as derived above. The number density thus allows for a double *bias expansion* in the matter density and potential

$$n(\nu) = \bar{n}(\nu) + \left.\frac{\partial n}{\partial \delta_{\rm l}}\right|_{\delta_{\rm l}=\varphi_{\rm G,l}=0}\delta_{\rm l} + \left.\frac{\partial n}{\partial \varphi_{\rm G,l}}\right|_{\delta_{\rm l}=\varphi_{\rm G,l}=0}\varphi_{\rm G,l} + \cdots ,$$

leading to the following expression for the galaxy overdensity:

$$\delta_{\rm g} = \frac{n}{\bar{n}} - 1 = b_{10}\delta_{\rm l} + b_{01}\varphi_{\rm G,l} + \cdots .$$

Calculate the coefficients b_{10} and b_{01} of this expansion in terms of $\nu = \delta_{\rm c}/\sigma_{\rm s}$ and express the non-Gaussian bias coefficient b_{01} in terms of the Gaussian bias coefficient b_{10}. Calculate the linear *galaxy* power spectrum $\langle\delta_{\rm g}\delta_{\rm g}\rangle$ and factor out the linear *matter* power spectrum $\langle\delta_{\rm G,l}\delta_{\rm G,l}\rangle$. Describe and sketch the behaviour of the power spectrum on large scales in the presence of primordial non-Gaussianity compared to the Gaussian case, considering both $f_{\rm NL} > 0$ and $f_{\rm NL} < 0$.

(iii) Calculate an explicit expression for the late-time tree-level *matter* density bispectrum in the presence of local-type primordial non-Gaussianity. Ignore the gravitational contribution to the bispectrum. Express your result in terms of the matter power spectrum and discuss its behaviour in the squeezed limit.

(iv) Based on the consistency condition for single-field inflation, explain the typical size of the bispectrum in the squeezed limit in such models. What does this imply for the behaviour of the galaxy power spectrum discussed in (ii) above?

2 The effective 1D dynamics of a set of parallel, infinite mass sheets in 3D space is described by the following continuity and Euler equations

$$\delta'(x,\tau) + \theta(x,\tau) = -\nabla\left[v(x,\tau)\delta(x,\tau)\right] ,$$
$$v'(x,\tau) + \mathcal{H}v(x,\tau) + \nabla\phi(x,\tau) = -v(x,\tau)\nabla v(x,\tau) ,$$

where δ is the overdensity, θ the velocity divergence ($\theta = \nabla v$) and the prime denotes the derivative with respect to conformal time τ. Assume a matter-only universe in which $\mathcal{H} = H_0 a^{-1/2}$ and $\nabla^2\phi(x,\tau) = \frac{3}{2}\mathcal{H}^2\delta(x,\tau)$. Rewriting the above equations in Fourier space, one obtains

$$\delta'(k,\tau) + \theta(k,\tau) = -\int\frac{\mathrm{d}q_1}{2\pi}\int \mathrm{d}q_2\alpha(q_1,q_2)\theta(q_1,\tau)\delta(q_2,\tau)\delta^{(\mathrm{D})}(k-q_1-q_2) ,$$
$$\theta'(k,\tau) + \mathcal{H}\theta(k,\tau) + \frac{3}{2}\mathcal{H}^2\delta(k) = -\int\frac{\mathrm{d}q_1}{2\pi}\int \mathrm{d}q_2\beta(q_1,q_2)\theta(q_1,\tau)\theta(q_2,\tau)\delta^{(\mathrm{D})}(k-q_1-q_2) ,$$

where $\alpha(q_1,q_2) = \beta(q_1,q_2) = (q_1+q_2)/q_1$.

(i) Consider the k-space versions of the continuity and Euler equations and the perturbative expansions of the density and velocity divergence fields

$$\delta(k,\tau) = \sum_{n=1}^{\infty} a^n(\tau)\prod_{m=1}^{n}\left\{\int\frac{\mathrm{d}q_m}{2\pi}\delta_0^{(1)}(q_m)\right\}F_n(q_1,\ldots,q_n)(2\pi)\delta^{(\mathrm{D})}\left(k-\sum_{m=1}^{n}q_m\right) ,$$
$$\theta(k,\tau) = -\mathcal{H}\sum_{n=1}^{\infty} a^n(\tau)\prod_{m=1}^{n}\left\{\int\frac{\mathrm{d}q_m}{2\pi}\delta_0^{(1)}(q_m)\right\}G_n(q_1,\ldots,q_n)(2\pi)\delta^{(\mathrm{D})}\left(k-\sum_{m=1}^{n}q_m\right) ,$$

where $\delta_0^{(1)}(k)$ is the linear, Gaussian random field normalized at $a = 1$ and $F_1(q_1) = 1$. Derive the gravitational coupling kernels G_1, F_2 and G_2. Show explicitly that the symmetrized version of the second-order gravitational kernel is given by $F_{2,\mathrm{s}}(q_1,q_2) = (q_1+q_2)^2/(2q_1q_2)$.

(ii) It can be shown that the functional form of $F_{2,\mathrm{s}}$ generalizes to higher order as

$$F_{n,\mathrm{s}}(q_1,\ldots,q_n) = \frac{(\sum_i q_i)^n}{n!\prod_i q_i} .$$

Draw the Feynman diagrams for the one-loop bispectra arising from *a*) the product of three second-order fields $\langle\delta^{(2)}\delta^{(2)}\delta^{(2)}\rangle$ and *b*) the product of two first-order fields and one fourth-order field $\langle\delta^{(1)}\delta^{(1)}\delta^{(4)}\rangle$. Write down the corresponding expressions for the contributions to the bispectra in terms of Fourier-space integrals over linear density power spectra. Discuss the behaviour of the diagrams if the loop momentum q becomes large compared to the external momenta. Which of the two diagrams has the leading UV-sensitivity for wavenumbers below the non-linear wavenumber k_{NL} and what would be an appropriate counterterm for the leading divergence at the field level? For this purpose, consider second spatial derivatives of the perturbative density fields.

3 (i) After neutrino decoupling, but while still relativistic, neutrinos can be described by the collisionless Boltzmann equation for massless particles. For scalar perturbations (in the conformal Newtonian gauge) about a spatially-flat universe, this takes the form

$$\dot{\Theta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta + \mathbf{e}\cdot\boldsymbol{\nabla}\psi - \dot{\phi} = 0\,.$$

Here, $\Theta(\eta, \mathbf{x}, \mathbf{e})$ is the dimensionless neutrino temperature perturbation at conformal time η and comoving position $\mathbf{x}$, the unit vector $\mathbf{e}$ is the neutrino propagation direction and ψ and ϕ are the metric perturbations. Overdots denote differentiation with respect to η. By expanding Θ in Fourier and angular modes as

$$\Theta(\eta, \mathbf{x}, \mathbf{e}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(\sum_{l\geqslant 0} (-i)^l \Theta_l(\eta, \mathbf{k}) P_l(\hat{\mathbf{k}}\cdot\mathbf{e}) \right) e^{i\mathbf{k}\cdot\mathbf{x}}\,,$$

where $P_l(\mu)$ are the Legendre polynomials and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, obtain the Boltzmann hierarchy

$$\dot{\Theta}_l + k\left(\frac{l+1}{2l+3}\Theta_{l+1} - \frac{l}{2l-1}\Theta_{l-1}\right) = \delta_{l0}\dot{\phi} + \delta_{l1}k\psi\,. \qquad (*)$$

[*You may wish to use* $(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)$ *and* $P_0(\mu) = 1$ *and* $P_1(\mu) = \mu$*.*]

(ii) The orthonormal-frame components of the neutrino anisotropic stress are

$$\Pi^{\hat{i}\hat{j}}(\eta, \mathbf{x}) = -4\bar{\rho}_\nu \int \frac{d\mathbf{e}}{4\pi}\,\Theta(\eta, \mathbf{x}, \mathbf{e})\left(e^{\hat{i}}e^{\hat{j}} - \frac{1}{3}\delta^{\hat{i}\hat{j}}\right)\,,$$

where $\bar{\rho}_\nu$ is the unperturbed neutrino energy density. Show that

$$\Pi^{\hat{i}\hat{j}}(\eta, \mathbf{x}) = -\frac{4}{3}\bar{\rho}_\nu \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\,\Pi(\eta, \mathbf{k})\left(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta^{ij}\right)e^{i\mathbf{k}\cdot\mathbf{x}}\,,$$

with $\Pi(\eta, \mathbf{k}) = -3\Theta_2(\eta, \mathbf{k})/5$.
[*You may wish to use* $P_2(\mu) = (3\mu^2 - 1)/2$ *and to note that*

$$\int \frac{d\mathbf{e}}{4\pi}\,P_l(\hat{\mathbf{k}}\cdot\mathbf{e})\left(e^{\hat{i}}e^{\hat{j}} - \frac{1}{3}\delta^{\hat{i}\hat{j}}\right) = A_l\left(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta^{ij}\right)$$

for some A_l*.*]

(iii) On super-Hubble scales in radiation domination for adiabatic initial conditions, ϕ and ψ are constant in time with values $\phi(0, \mathbf{k})$ and $\psi(0, \mathbf{k})$, respectively. Assuming that $\Theta_0 = -\psi(0, \mathbf{k})/2 + O(k\eta)$ and $\Theta_l = O\left((k\eta)^l\right)$, use the $l = 1$ and $l = 2$ moments of $(*)$ to show that at leading order

$$\Theta_1 = \frac{1}{2}\psi(0, \mathbf{k})k\eta \quad \text{and} \quad \Theta_2 = \frac{1}{6}\psi(0, \mathbf{k})(k\eta)^2\,.$$

(iv) The i-j Einstein equation gives

$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{1}{3}\delta_{ij}\nabla^2\right)(\phi - \psi) = -8\pi G a^2 \Pi_{ij}\,,$$

where a is the scale factor and, numerically, $\Pi_{ij} = \Pi^{\hat{\imath}\hat{\jmath}}$ at first order in perturbations. Assuming neutrinos are the only source of anisotropic stress, show that

$$\phi(0,\mathbf{k}) = \left(1 + \frac{2}{5} f_\nu\right) \psi(0,\mathbf{k}) \,,$$

where $f_\nu = \bar{\rho}_\nu/(\bar{\rho}_\nu + \bar{\rho}_\gamma)$ with $\bar{\rho}_\gamma$ the unperturbed photon energy density.

The comoving-gauge curvature perturbation can be expressed as

$$\mathcal{R} = -\phi - \frac{\mathcal{H}(\dot{\phi} + \mathcal{H}\psi)}{4\pi G a^2(\bar{\rho} + \bar{P})} \,,$$

where $\mathcal{H} = \dot{a}/a$ and $\bar{\rho}$ and $\bar{P}$ are the unperturbed total energy density and pressure, respectively. Given that $\mathcal{R}$ is constant in time, with value $\mathcal{R}(\mathbf{k})$, on super-Hubble scales for adiabatic initial conditions, express $\phi(0,\mathbf{k})$ and $\psi(0,\mathbf{k})$ in terms of $\mathcal{R}(\mathbf{k})$.

4 (i) The optical depth to Thomson scattering between conformal time η and the present (time η_0) is

$$\tau(\eta) = \int_\eta^{\eta_0} a\bar{n}_e \sigma_{\mathrm{T}}\, d\eta' \,,$$

where a is the scale factor, $\bar{n}_e$ is the unperturbed electron number density and σ_{T} is the Thomson cross-section. What are the physical interpretations of $e^{-\tau}$ and the visibility function $g(\eta) = -(d\tau/d\eta)e^{-\tau}$? Sketch these quantities from times well before recombination to the present assuming that the universe does not reionize after recombination.

(ii) The dimensionless temperature anisotropy of the CMB is $\Theta(\eta, \mathbf{x}, \mathbf{e})$, where $\mathbf{x}$ is comoving position and $\mathbf{e}$ is the photon propagation direction. For linear scalar perturbations about a spatially-flat universe, the Boltzmann equation for Θ in the conformal Newtonian gauge is

$$\dot{\Theta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta + \mathbf{e}\cdot\boldsymbol{\nabla}\psi - \dot{\phi} = -a\bar{n}_e\sigma_{\mathrm{T}}\Theta + \frac{3a\bar{n}_e\sigma_{\mathrm{T}}}{16\pi}\int d\hat{\mathbf{m}}\,\Theta(\hat{\mathbf{m}})\left[1 + (\mathbf{e}\cdot\hat{\mathbf{m}})^2\right] + a\bar{n}_e\sigma_{\mathrm{T}}\mathbf{e}\cdot\mathbf{v}_b \,,$$

where ϕ and ψ are the metric perturbations, $\mathbf{v}_b$ is the baryon peculiar velocity and overdots denote partial differentiation with respect to η. Stating carefully any assumptions you make, show that the temperature anisotropy observed at $(\eta_0, \mathbf{x}_0)$ from direction $\mathbf{e}$ satisfies

$$\Theta(\eta_0, \mathbf{x}_0, \mathbf{e}) + \psi(\eta_0, \mathbf{x}_0) \approx \int_0^{\eta_0} g(\eta')\,(\Theta_0 + \psi + \mathbf{e}\cdot\mathbf{v}_b)\,(\eta', \mathbf{x}_0 - \chi\mathbf{e})\,d\eta' + \int_0^{\eta_0} e^{-\tau}\left(\dot{\phi} + \dot{\psi}\right)(\eta', \mathbf{x}_0 - \chi\mathbf{e})\,d\eta' \,, \quad (*)$$

where $\chi = \eta_0 - \eta'$ and $\Theta_0 = \int d\hat{\mathbf{m}}\,\Theta(\hat{\mathbf{m}})/(4\pi)$ is the monopole of Θ.

Simplify $(*)$ in the limit of instantaneous last scattering at time η_* and assuming the universe is matter dominated between η_* and the present. Give a physical interpretation of the various terms in your expression for $\Theta(\eta_0, \mathbf{x}_0, \mathbf{e})$.

(iii) In a model where the universe reionized at time η_{re}, the visibility function can be approximated by

$$g(\eta) \approx \left(1 - e^{-\tau_{\mathrm{re}}}\right)\delta(\eta - \eta_{\mathrm{re}}) + e^{-\tau_{\mathrm{re}}}\delta(\eta - \eta_*) \,,$$

where τ_{re} is the optical depth back to η_{re}. Assuming a matter-dominated universe for $\eta \geqslant \eta_*$ and ignoring the effects of $\mathbf{v}_b$, apply $(*)$ twice to show that

$$\Theta(\eta_0, \mathbf{x}_0, \mathbf{e}) + \psi(\eta_0, \mathbf{x}_0) \approx e^{-\tau_{\mathrm{re}}}\,(\Theta_0 + \psi)\,(\eta_*, \mathbf{x}_0 - \chi_*\mathbf{e}) + \left(1 - e^{-\tau_{\mathrm{re}}}\right)\int \frac{d\hat{\mathbf{m}}}{4\pi}\,(\Theta_0 + \psi)\,(\eta_*, \mathbf{x}_0 - \chi_{\mathrm{re}}\mathbf{e} - \Delta\chi\hat{\mathbf{m}}) \,,$$

where $\chi_* = \eta_0 - \eta_*$, $\chi_{\mathrm{re}} = \eta_0 - \eta_{\mathrm{re}}$ and $\Delta\chi = \chi_* - \chi_{\mathrm{re}}$.
[*Hint: you should consider how* $\Theta_0 + \psi$ *at* $(\eta_{\mathrm{re}}, \mathbf{x}_0 - \chi_{\mathrm{re}}\mathbf{e})$ *is related to the fluctuations at time* η_*.]

Hence show that for perturbations with coherence length $1/k$,

$$\Theta(\eta_0, \mathbf{x}_0, \mathbf{e}) + \psi(\eta_0, \mathbf{x}_0) \approx \begin{cases} e^{-\tau_{\mathrm{re}}}\,(\Theta_0 + \psi)\,(\eta_*, \mathbf{x}_0 - \chi_*\mathbf{e}) & \text{for } k\Delta\chi \gg 1 \,, \\ (\Theta_0 + \psi)\,(\eta_*, \mathbf{x}_0 - \chi_*\mathbf{e}) & \text{for } k\Delta\chi \ll 1 \,. \end{cases}$$

Explain why observations of the CMB temperature anisotropies measure the parameter combination $A_{\rm s}e^{-2\tau_{\rm re}}$, where $A_{\rm s}$ is the amplitude of the primordial power spectrum, much more precisely than either of $A_{\rm s}$ or $\tau_{\rm re}$ individually.

END OF PAPER