UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 9:00 am to 12:00 pm

PAPER 311

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In a globally hyperbolic spacetime one can introduce coordinates (t, x^i) such that

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)\,.$$

The Einstein-Hilbert action is then (neglecting surface terms and setting $16\pi G = 1$)

$$S = \int \mathrm{d}t\, \mathrm{d}^3x\, \sqrt{h}\, N({}^{(3)}R + K_{ij}K^{ij} - K^2)$$

where ${}^{(3)}R$ is the Ricci scalar of h_{ij}, h the determinant of h_{ij} and

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - D_i N_j - D_j N_i)\,, \quad K = h^{ij} K_{ij}$$

where D_i is the Levi-Civita connection of h_{ij}.

(i) Why are N and N^i non-dynamical fields? Determine the momentum π^{ij} conjugate to h_{ij}.

(ii) Show that the Hamiltonian of General Relativity is (neglecting surface terms)

$$H = \int \mathrm{d}^3x\, \sqrt{h}\, (N\, \mathcal{H} + N^i \mathcal{H}_i)$$

where $\mathcal{H}$ and $\mathcal{H}_i$ should be expressed in terms of h_{ij} and π_{ij}.

(iii) Consider the case in which surfaces of constant t are each asymptotically flat with one end. Using Hamilton's equations

$$\partial_t h_{ij} = \frac{\delta H}{\delta \pi^{ij}}\,, \qquad \partial_t \pi^{ij} = -\frac{\delta H}{\delta h_{ij}}$$

explain why a well defined variational problem requires adding to H a term of the following form

$$E_{\mathrm{ADM}} = \lim_{r \to +\infty} \int_{S_r^2} \mathrm{d}A\, n^i\, (\partial_j h_{ij} - \partial_i h_{jj})\,,$$

where S_r^2 is a two-sphere of radius r, $\mathrm{d}A$ is the area element of S_r^2 and n^i its outward unit normal.

[You may use

$$\delta^{(3)}R = -{}^{(3)}R^{ij} \delta h_{ij} + D^i D^j \delta h_{ij} - D^k D_k (h^{ij} \delta h_{ij})\,.]$$

(iv) Why does a closed universe have zero energy?

2

(a) Consider a null geodesic congruence that contains the generators of a Killing horizon $\mathcal{N}$. Show that the expansion θ, shear $\hat{\sigma}$ and rotation $\hat{\omega}$ vanish on $\mathcal{N}$. *[You may assume Frobenius' theorem.]*

(b) State and prove the version of the first law of black hole mechanics that relates the change in area of the event horizon to the energy and angular momentum of infalling matter. *[You may assume Raychaudhuri's equation.]*

(c) Consider the Penrose process for a Kerr black hole. By considering the 4−momentum, explain why the particle that falls through the horizon has energy E and angular momentum L obeying $E \geqslant \Omega\, L$, where Ω is the black hole angular velocity. Show that the same result can be obtained from the first and second laws of black hole mechanics.

3

The following four-dimensional black hole spacetime

$$\mathrm{d}s^2 = -V(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + R(r)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) , \quad A = P\cos\theta\,\mathrm{d}\varphi , \quad \phi(r) = -\frac{1}{2}\log\left(1 - \frac{r_-}{r}\right) ,$$

where

$$R^2(r) = r\,(r - r_-) , \quad r_- = \frac{2P^2}{r_+} \quad \text{and} \quad V(r) = 1 - \frac{r_+}{r} ,$$

is a solution to the equations of motion derived from the following action

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left(R - 2\nabla^a\phi\nabla_a\phi - e^{-2\phi}F^{ab}F_{ab}\right) ,$$

where $F = \mathrm{d}A$ and P is a constant magnetic flux. You may assume that

$$R^{abcd}R_{abcd} = \frac{3r^2r_-^4 - 2r\left(8r^2 - 10r_-r + 5r_-^2\right)r_+r_-^2 + 3\left(4r^2 - 6r_-r + 3r_-^2\right)^2 r_+^2}{4r^6\left(r - r_-\right)^4} .$$

(i) Calculate the Komar mass M_{Komar} of this solution defined by

$$M_{\mathrm{Komar}}(r) = -\frac{1}{8\pi}\int_{S_r^2} \star\,\mathrm{d}k ,$$

where $k^a = (\partial/\partial t)^a$. The integral is taken over a constant $t\,,r$ surface and the orientation is $\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi$. Explain why $M_{\mathrm{Komar}}(r)$ can depend on r if $R_{ab} \neq 0$.

(ii) Show that the geodesic equation for null, spacelike and timelike geodesics can be reduced to an equation of the form

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \tilde{V}(r) = 0 ,$$

where τ is a suitable affine parameter. Determine $\tilde{V}(r)$.

(iii) Show that one can define a quantity $r_\star$ such that $u = t - r_\star$ and $v = t + r_\star$ are constant on radial outgoing and ingoing null geodesics, respectively.

(iv) Define the *black hole region* of an asymptotically flat spacetime. Prove that if $r_- < r_+$ then the region $r_- < r < r_+$ is within the black hole region. Sketch the Penrose diagram for $r_- < r_+$. What does the spacetime describe when $r_- > r_+$?

(v) Assuming $r_- < r_+$, show that the surface $r = r_+$ is a Killing horizon of k and determine the surface gravity κ.

4 Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime and introduce coordinates (t, x^i) such that

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)\,.$$

Consider a massive real scalar field Φ with action

$$S = \int_{\mathcal{M}} \mathrm{d}t\, \mathrm{d}^3x \sqrt{-g}\left(-\frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi - \frac{1}{2}m^2\Phi^2\right)$$

and equation of motion

$$g^{ab}\nabla_a\nabla_b\Phi - m^2\Phi = 0\,.$$

(i) Let $\mathcal{S}$ be the space of complex solutions of the Klein-Gordon equation endowed with the inner product

$$(\alpha, \beta) = i\int_{\Sigma_t} \mathrm{d}^3x\sqrt{h}\, n^a\left(\bar{\alpha}\nabla_a\beta - \beta\nabla_a\bar{\alpha}\right),$$

where Σ_t denotes a Cauchy surface of constant t and n_a its future-directed unit normal. Show that (α, β) is independent of Σ_t and that $(\alpha, \beta) = \overline{(\beta, \alpha)}$.

(ii) In Minkowski space, $(,)$ is positive definite on the subspace $\mathcal{S}_p$ of $\mathcal{S}$ consisting of positive frequency solutions. Show that the positive frequency plane waves

$$\psi_{\mathbf{p}} = \frac{1}{(2\pi)^{3/2}(2p^0)^{1/2}}e^{ip\cdot x} \qquad \text{with} \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2}\,,$$

form a basis for $\mathcal{S}_p$ with respect to inertial coordinates $(t, \mathbf{x})$ and explain in which sense these have positive frequency.

(iii) Consider again a general globally hyperbolic spacetime. Let $\{\psi_i\}$ be an orthonormal basis for a choice of positive frequency subspace $\mathcal{S}_p$ of $\mathcal{S}$:

$$(\psi_i, \psi_j) = \delta_{ij}\,, \qquad (\psi_i, \bar{\psi}_j) = 0\,.$$

A quantum field $\Phi(x)$ takes the form

$$\Phi(x) = \sum_i \left(a_i\psi_i + a_i^\dagger\bar{\psi}_i\right),$$

where the operator coefficients a_i and $a_i^\dagger$ satisfy the commutation relations

$$[a_i, a_j] = 0\,, \qquad [a_i, a_j^\dagger] = \delta_{ij}.$$

Let $\mathcal{S}'_p$ be a different choice for the positive frequency subspace of $\mathcal{S}$, with orthonormal basis $\{\psi'_i\}$ given by

$$\psi'_i = \sum_j (A_{ij}\psi_j + B_{ij}\bar{\psi}_j).$$

(a) Show that

$$a'_i = \sum_j (\bar{A}_{ij}a_j - \bar{B}_{ij}a_j^\dagger)\,.$$

(b) Find the restrictions on the matrices A and B that follow from the requirement that the primed operators obey the same commutation relations as the unprimed operators.

(c) Consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ which is Minkowski in the far past and in the far future. Show that if the state is the vacuum for inertial observers at early times then it will contain particles for inertial observers at late times unless $B = 0$.

END OF PAPER