UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 1:30 pm to 4:30 pm

PAPER 310

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(a)** Our assumptions of homogeneity and isotropy force the background metric describing the universe to take the form,

$$ds^2 = -dt^2 + b^2(t)d\ell^2\,,$$

where $b(t)$ is an arbitrary function of time and $d\ell^2$ is a constant curvature three metric. If we assume the curvature is positive, so the space is spherical, then the 3-metric takes the form,

$$d\ell^2 = d\mathbf{x}^2 + du^2 \quad \text{with constraint} \quad |\mathbf{x}|^2 + u^2 = R^2\,,$$

where R is the radius of curvature of the space.

(i) Show this leads to a metric of the form,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)\,,$$

where you should define $a(t)$ and K.

Our assumptions of homogeneity and isotropy also lead us to assume that the energy momentum tensor takes the form of a perfect fluid, and with the form of the metric above, the Einstein equations reduce to the Friedmann and acceleration equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \qquad\qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right)\,.$$

These equations can be combined to obtain the continuity equation,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P\right)\,,$$

but while the Friedman equations only hold for the total energy density and total pressure, $\rho = \sum_i \rho_i$, $P = \sum_i P_i$, the continuity equation holds for each fluid component separately.

(ii) Explain why the continuity equation holds for each fluid component separately, unlike the Friedman equations.

(b) Consider a closed universe where in addition to the usual matter, radiation and dark energy components we also have a non-interacting cosmic string network, whose energy density scales as a^{-2}, and a non-interacting domain wall network, whose energy density scales as a^{-1}.

(i) Find the effective equation of state for strings and also that for walls.

(ii) Define the fractional densities, $(\Omega_{r,0}, \Omega_{m,0}, \Omega_{s,0}, \Omega_{w,0}, \Omega_{\Lambda,0})$, at time $t = t_0$, and give the Friedmann and acceleration equations in terms of them.

(iii) Find a static solution for $a(t)$ in terms of a single component.

(iv) Demonstrate the stability, or otherwise, of the static solution with respect to changes in the scale factor. What happens if instead we slightly perturb the energy density?

2 **(a)** We define the field $\delta_R(\mathbf{x})$ by smoothing the matter perturbation, $\delta(\mathbf{x})$, via convolution with a window function $W(r, R)$. The field $\delta_R(\mathbf{x})$ is then a Gaussian random field with probability distribution function,

$$P(\delta_R, \sigma_R) = \frac{1}{\sqrt{2\pi}\sigma_R} \exp\left(-\frac{\delta_R^2}{2\sigma_R^2}\right), \qquad \text{where} \qquad \sigma_R^2 = \frac{1}{2\pi^2} \int_0^\infty P(k)\widetilde{W}^2(kR)k^2 dk\,.$$

$P(k)$ is the power spectrum of δ_R and $\widetilde{W}(kR)$ is the Fourier transform of the window function $W(r, R)$.

(i) Assuming $P(k) \propto k^{n_{eff}}$, where n_{eff} is constant and $n_{eff} < 3$. Show that

$$\sigma_R \propto R^{-\left(\frac{n_{eff}+3}{2}\right)}\,.$$

In the Press-Schechter formalism we associated a halo with a location in space, $\mathbf{x}$, by choosing the largest smoothing scale, R, for which $\delta_R(\mathbf{x})$ exceeds a critical value, $\delta_c = 1.69$. The monotonic inverse relationship between the smoothing scale R and the variance $S = \sigma_R^2$ allowed us to use S to label the smoothing scale. At $S = 0$ (corresponding to $R \to \infty$) we must have $\delta_S(\mathbf{x}) = 0$. As we increase S (decrease R), locally $\delta_S(\mathbf{x})$ will perform a random walk with S playing the role of time. Thus we can equivalently associate a halo with a point x by finding the smallest S for which the random walk, $\delta_S(\mathbf{x})$, crosses the barrier δ_c.

(ii) Calculate the probability that $\delta_S(\mathbf{x}) \geqslant \delta_c$ for a given $S = S_*$ at the location x (i.e. the random walk is above the barrier δ_c at S_*)? Your answer may be in integral form.

(iii) Explain the mirror trajectories argument given in lectures and the key assumption required for the smoothing window function. Calculate the probability that $\delta_S(\mathbf{x}) \geqslant \delta_c$ for any $S \leqslant S_*$ at the location x (i.e. the random walk crossed the barrier δ_c before S_*)?

(iv) The mass function is defined $\frac{d\bar{n}_h}{dM} \equiv -\frac{1}{V_M}\frac{dP}{dM}$, where M is the mass of the halo, P is the probability of forming a halo with mass greater than M per unit volume, V_M is the volume needed to form a halo with mass M. Use your previous results to show that the mass function is,

$$\frac{d\bar{n}_h}{dM} = -\sqrt{\frac{2}{\pi}} \frac{\bar{\rho}}{M\sigma_R} \frac{d\sigma_R}{dM} \nu \exp\left(-\frac{\nu^2}{2}\right),$$

where you should define ν. [*You may use the following:* $2\int_a^b y^2 e^{-y^2} dy = \int_a^b e^{-y^2} dy - |ye^{-y^2}|_a^b$]

(b) In warm dark matter models structure formation is suppressed on small scales leading to a cut-off in the power spectrum,

$$P(k) \propto \begin{cases} k^{n_{eff}} & (k < k_{WDM}) \\ 0 & (k > k_{WDM}) \end{cases}.$$

How does this affect the arguments used in part **(a)**? Can we still use the mass function we derived?

3

(a) From the first law of thermodynamics, show that the entropy S in a comoving volume V is given by

$$S = \frac{\rho + P}{T} V \,,$$

where P is the pressure and ρ is the energy density. Show also that the entropy is conserved in equilibrium in an expanding universe. You may assume that the chemical potential is zero in both calculations.

[*Hint: You are free to use the result* $\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$*, the first law of thermodynamics* $dU = TdS - PdV + \mu dN$*, and the continuity equation* $\dot{\rho} + (\rho + P)\frac{\dot{V}}{V} = 0$.]

(b) Starting from your result in **(a)**, show that a quantity $g_{*s} a^3 T^3$ is conserved in the early universe, where you should define g_{*s}. Explain in detail why this implies that the present-day temperature of the cosmic neutrino background is related to the temperature of cosmic microwave background photons by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

(c) Consider a hot big bang universe consisting of the standard model and, in addition, a non-relativistic, non-baryonic particle χ. Any interactions of χ with other existing particles can be neglected for the purposes of this question. At a time t_d, which is after electron-positron annihilation but early enough that the universe's energy is radiation-dominated, the χ particles decay instantaneously into photons. These photons rapidly thermalize with the existing photon background. Assuming that energy is conserved in this instantaneous decay, show that the ratio of photon to neutrino background temperatures is modified to

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \left(1 + \left(\frac{t_d}{t_{\mathrm{ref}}}\right)^{1/2} \frac{\rho_\chi(t_{\mathrm{ref}})}{\rho_\gamma(t_{\mathrm{ref}})}\right)^{-1/4} .$$

Here ρ_χ is the energy density of the χ particles, ρ_γ is the energy density in photons, and t_{ref} is a reference time after electron-positron annihilation but before t_d at which ρ_χ and ρ_γ are specified.

[*Hint: You may assume that in radiation domination, the scale factor of the universe evolves as* $a \propto t^{1/2}$]

(d) By what factor does the baryon-to-photon ratio η change due to the decay of χ particles? In light of your result and the fact that the current value of η is already well determined from the CMB, explain briefly how precise measurements of light element abundances from big bang nucleosynthesis could be used to find evidence for a decay of χ.

4

Single-field slow-roll inflation is described by the scalar field action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right],$$

where ϕ is the inflation field and $V(\phi)$ is its potential. Throughout the question, you may assume that you can ignore metric fluctuations and that the space-time is described by the line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2$, or equivalently $\mathrm{d}s^2 = a(\tau)^2\left[-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right]$ when written in terms of conformal time τ. You may also assume throughout the entire problem that $a(\tau) = -(H\tau)^{-1}$, that $H = \sqrt{\frac{V(\phi)}{3Mpl^2}} \approx$ constant, and that $2H^2 \gg \partial^2_{\phi\phi}V(\phi)$.

(a) Show that the equation of motion for the field ϕ is

$$\phi'' + 2\frac{a'}{a}\phi' - \nabla^2\phi + a^2\frac{\partial}{\partial\phi}V(\phi) = 0,$$

where $'$ indicates a derivative with respect to conformal time.

(b) Consider a perturbation of the field $\phi(\tau,\mathbf{x}) = \bar{\phi}(\tau) + f(\tau,\mathbf{x})/a(\tau)$. Neglecting metric fluctuations, show that each Fourier mode of the perturbation obeys the following equation

$$f''_{\mathbf{k}} + (k^2 - \frac{a''}{a})f_{\mathbf{k}} = 0.$$

Why does canonical quantization of $f_{\mathbf{k}}$ proceed similarly to the quantization of a simple harmonic oscillator?

(c) Canonical quantization leads to the following expression for the field operator:

$$\hat{f}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[f_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}.\mathbf{x}} + f^*_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}.\mathbf{x}}\right],$$

where $f^*_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$, $k = |\mathbf{k}|$, and $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}$ are raising and lowering operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}'}$. Calculate the two point correlation function of $\delta\phi$ at different positions (using $\hat{\delta\phi}(\tau,\mathbf{x}) = \hat{f}(\tau,\mathbf{x})/a(\tau)$),

$$\langle 0|\hat{\delta\phi}(\tau,\mathbf{x})\hat{\delta\phi}(\tau,\mathbf{x}+\mathbf{r})|0\rangle.$$

Deduce the dimensionless power spectrum of $\delta\phi$, evaluate it when $k \ll aH$, and show that the spectrum is nearly scale invariant in this limit. Discuss briefly the significance of this result.

[*Hint: you may assume that the dimensionless power spectrum* $\Delta^2_{\delta\phi}$ *is related to the two point correlation function via* $\langle 0|\hat{\delta\phi}(\tau,\mathbf{x})\hat{\delta\phi}(\tau,\mathbf{x}+\mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{2\pi^2}{k^3}\Delta^2_{\delta\phi}e^{-i\mathbf{k}.\mathbf{r}}$]

(d) Assuming that the expression for the field operator in **(c)** can still be used for $\hat{\delta\phi}$, evaluate the three point correlation function $\langle 0|\hat{\delta\phi}(\tau,\mathbf{x})\hat{\delta\phi}(\tau,\mathbf{y})\hat{\delta\phi}(\tau,\mathbf{z})|0\rangle$ for three arbitrary coordinates $\mathbf{x},\mathbf{y},\mathbf{z}$. Under the same assumptions, calculate a general expression for all odd moments of the field $\langle 0|(\hat{f}(\tau,0))^{2n+1}|0\rangle$ (where n is an integer).

END OF PAPER