UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2018 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\}/\sim$, where

$$[X_1, X_2, \ldots, X_{n+1}] \sim [cX_1, cX_2, \ldots, cX_{n+1}], \quad c \in \mathbb{R} \setminus \{0\}.$$

Show, by constructing an atlas consisting of $(n+1)$ charts that $\mathbb{RP}^n$ is an n–dimensional manifold.

(b) Let $U \subset \mathbb{R}^n$ be an open set, and let $\pi : U \to \mathbb{R}^{n+1} \setminus \{0\}$ be given by $\pi(y_1, \ldots, y_n) = [y_1, \ldots, y_n, 1]$. Construct an explicit expression for the Riemannian metric g on U such that

$$g = \pi^* G, \quad \text{where} \quad G = \frac{dX_1^2 + dX_2^2 + \cdots + dX_{n+1}^2}{X_1^2 + X_2^2 + \cdots + X_{n+1}^2}$$

is a Riemannian metric on $\mathbb{R}^{n+1} \setminus \{0\}$.

(c) Assume that $n = 2$, and compute the Ricci scalar of g in the coordinates (y_1, y_2).

[*Hint: If (M, g) is a Riemannian manifold of dimension N, and $\Omega : M \to \mathbb{R}^+$ then*

$$\widetilde{R} = \Omega^{-2}\Big(R - 2(N-1)g^{ab}\nabla_a\nabla_b \ln\Omega - (N-2)(N-1)g^{ab}(\nabla_a \ln\Omega)(\nabla_b \ln\Omega)\Big)$$

where R and ∇ are respectively the Ricci scalar and the Levi–Civita connection of g, and $\widetilde{R}$ is the Ricci scalar of $\tilde{g}_{ab} = \Omega^2 g_{ab}$.]

2

Let (M, g) be a (pseudo) Riemannian manifold.

(i) Let $R^d{}_{abc}$ be the curvature tensor of the Levi–Civita connection ∇ of g. Show that

$$R^d{}_{[abc]} = 0.$$

[*Hint. You may assume the existence of normal coordinates.*]

(ii) Let K_a be a solution to the Killing equation $\nabla_{(a}K_{b)} = 0$. Show that

$$\nabla_a \nabla_b K^c = R^c{}_{bad} K^d, \qquad (1)$$

and demonstrate that $K_a V^a$ is constant along an affinely parametrised geodesic with tangent vector V^a.

(iii) Use (1) to show that the most general Killing vector on $M = \mathbb{R}^n$ where g is the flat Euclidean metric is of the form

$$K = A^\mu \frac{\partial}{\partial x^\mu} + M_\mu{}^\nu x^\mu \frac{\partial}{\partial x^\nu} \qquad (2)$$

where $A \in \mathbb{R}^n$ is a constant vector, and $M_{\mu\nu} \equiv M_\mu{}^\kappa \delta_{\kappa\nu}$ is a constant matrix such that $M^T = -M$.

(iv) Assume that A is general, $M_{21} = -M_{12} = 1$, and all other components of M vanish. Find the integral curves of the vector field (2).

3

(a) Starting from the linearized Einstein equation $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$ in harmonic gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$, show that, far from a non-relativistic matter distribution

$$\bar{h}_{ij}(t, \mathbf{x}) \approx \frac{2}{r} \ddot{I}_{ij}(t - r)$$

where $i, j = 1, 2, 3,\ r = |\mathbf{x}|$, and

$$I_{ij}(t) = \int d^3x\, T_{00}(t, \mathbf{x}) x^i x^j .$$

State clearly any assumptions that you make.

(b)(i) A rigid body has energy density $T_{00} = \rho(\mathbf{x})$. When the body is at rest, the spatial coordinates can be chosen so that $I_{ij} = \bar{I}_{ij}$ where $\bar{I}_{ij}$ is diagonal with components (I_{11}, I_{22}, I_{33}). Suppose that such a body rotates with angular velocity Ω about the x^3-axis so that $T_{00}(t, \mathbf{x}) = \rho(\mathbf{L}(t)^{-1}\mathbf{x})$ where $\mathbf{L}(t)$ is the 3×3 matrix describing a rotation through angle Ωt about the x^3-axis. Show that $I_{ij}(t) = L_{ik}(t) L_{jl}(t) \bar{I}_{kl}$ and hence determine the components of $I_{ij}(t)$.

(ii) What is the frequency of the gravitational waves emitted by the rotating body?

(iii) Show that the average power emitted in gravitational waves by the rotating body is

$$\langle P \rangle = \frac{32}{5} \Omega^6 \left(I_{11} - I_{22}\right)^2 .$$

4

(a) For a variation of the metric $g_{ab} \to g_{ab} + \delta g_{ab}$ derive formulae for the variations of the volume form, the inverse metric, and the Christoffel symbols. Show that the variation of the Ricci scalar can be written

$$\delta R = -R^{ab}\delta g_{ab} + \alpha \nabla^c \nabla_c \left(g^{ab} \delta g_{ab} \right) + \beta \nabla^a \nabla^b \delta g_{ab}$$

for coefficients α and β to be determined.

[*In a coordinate basis* $R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma}$]

(b)

(i) A real scalar field Φ has action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - \xi R \Phi^2 \right)$$

where ξ is a constant. Determine the energy momentum tensor of the scalar field.

(ii) Explain why this energy momentum tensor is conserved when the equation of motion of the scalar field is satisfied.

END OF PAPER