UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(i)** The phase-space action for an open Nambu-Goto string of tension T in a D-dimensional Minkowski spacetime with coordinates $\{X^m; m = 0, 1, \ldots, D-1\}$ is

$$I[X,P;e,u] = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (TX')^2 \right] - u\,(X^m)' P_m \right\}, \qquad (*)$$

where $e(t,\sigma)$ and $u(t,\sigma)$ are Lagrange multipliers. Where does the Minkowski spacetime metric appear in this action? Explain *briefly* why not all of the canonical variables are physical, and show how the action can be rewritten in terms of "transverse" phase space variables by imposing the Monge gauge conditions $X^0 = t$ and $X^1 = \sigma$. Use your result to explain why the tension T is also the rest-energy density of the string.

(ii) Explain *briefly* why the action $(*)$ is equivalent to the Polyakov action:

$$I_{\text{Poly}}[X;\gamma] = -\frac{T}{2} \int dt \int d\sigma \sqrt{-\det\gamma}\, \gamma^{\mu\nu} \partial_\mu X \cdot \partial_\nu X\,.$$

Show how the independent Polyakov metric $\gamma_{\mu\nu}$ of this action is related, by its equation of motion, to the worldsheet metric $g_{\mu\nu}$ induced by the Minkowski spacetime metric. Use your result to show that the Polyakov action is equivalent to the Nambu-Goto action $I_{NG}[X]$, which you should express in terms of the induced metric.

(iii) State the conformal gauge condition on the Polyakov metric and use it to find the Polyakov action in conformal gauge. Allowing for string endpoints, compute the variation of this action due to an arbitrary variation of X^m. Hence find the conformal gauge equation of motion for X^m, and show that the action is stationary at solutions of these equations if, *at each endpoint*, **either** X is fixed **or** $X' = 0$.

(iv) Explain why the Nambu-Goto equations for a string in a three-dimensional Minkowski spacetime ($D = 3$) are solved by the following configuration (for constant L):

$$X^0 = t\,, \qquad X^1 + iX^2 = Le^{it/L} \sin(\sigma/L)\,.$$

Given that this configuration describes an open string with one end fixed to the point $\vec{X} = \vec{0}$ and the other end free, what is its parameter length (range of σ)? What is its proper length? Explain why the string is rigidly rotating. What is its angular velocity of rotation; use your answer to determine the velocity of the free end of the string. Is your answer consistent with the boundary conditions at this end of the string? Find the total energy E of the string. How much of it is rotational energy? Find the total angular momentum J and show that $J = \beta E^2$ for some constant β, which you should determine.

2 **(i)** In a D-dimensional Minkowski space, with coordinates $\{x^m; m = 0, 1, \dots D-1\}$, the Fourier-mode form of the action for an open string with free ends is

$$I = \int dt \Big\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{j \in \mathbb{Z}} \lambda_{-j} L_j \Big\} . \qquad (*)$$

Explain briefly the physical meaning of the canonical variables of this action. Which are real and which are complex, and how is your answer compatible with reality of the action? What are the non-zero canonical Poisson bracket relations?

Write down an expression for L_j in terms of the canonical variables [*You may use the fact that* $\alpha_0^m = p^m/\sqrt{\pi T}$ *for a string of tension* T]. Write down (without proof) the Poisson bracket relations of the phase-space functions L_j; what can you conclude from them?

(ii) In (a version of) the light-cone gauge, the action $(*)$ reduces to

$$I = \int dt \Big\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - e_0 (p^2 + 2\pi T N) \Big\} .$$

Write down an expression for N in terms of the 'transverse' $(D-2)$-vector variables $\boldsymbol{\alpha}_k$. What are the canonical Poisson bracket relations now? Write down the canonical commutation relations and define the Fock vacuum state $|0\rangle$. Explain how a basis of Fock space states can be organized according to the eigenvalues N of a "level number" operator $\hat{N}$, defined such $\hat{N}|0\rangle = 0$. Write down the level-1 states and explain why they are polarization states of a *massless* particle. What does this imply for the ground state of the string?

(iii) Write down the canonical commutation relations corresponding to the canonical Poisson bracket relations of the action $(*)$, and define the 'covariant' Fock vacuum state $|0\rangle$. Show that there exists an operator ordering such that $\hat{L}_0 = p^2/(2\pi T) + \hat{N}_{\rm cov}$ where $\hat{N}_{\rm cov}|0\rangle = 0$. Write down (*without proof*) the commutation relations of the $\hat{L}_j$ and explain why they cannot be deduced directly from the Poisson bracket relations of the L_j. Why is no state of the string annihilated by $\hat{L}_j$ for all non-zero j?

Given that physical states satisfy (for constant a)

$$(\hat{L}_0 - a)|{\rm Phys}\rangle = 0 ; \qquad \hat{L}_j |{\rm Phys}\rangle = 0 , \quad j > 0 , \qquad (\dagger)$$

show that $|0\rangle$ is physical if $p^2 = (2\pi T)a$. Explain how a basis of states can be organized by a level number, as for the lightcone gauge, and why all level-1 states take the form $A_m \alpha_{-1}^m |0\rangle$ for some (p-dependent) coefficients A_m. Which of them satisfy $(\dagger)$?

Use your results to show that there is a level-1 state of negative norm if $a > 1$. Why do the level-1 states correspond to polarization states of a massive vector field when $a < 1$? Explain briefly why the spectrum of first excited states agrees with your light-cone gauge result if $a = 1$.

3 **(a)** The following action describes a relativistic point particle of mass M in a D-dimensional Minkowski spacetime ($m = 0, 1, \ldots D-1$):

$$I[x,p;e] = \int dt\, \{\dot{x}^m p_m - e\varphi\}\,, \qquad \varphi = \frac{1}{2}(p^2 + M^2)\,.$$

Write down the gauge transformation generated by the constraint function φ, and deduce the gauge transformation of $e(t)$ required for invariance of the action. Explain why the gauge-fixing condition $e(t) = s$ may be imposed, for 'variable' constant s, but also why the condition $e(t) = 1$ is too strong.

Given that gauge fixing is implemented in the path integral by insertion of the delta-functional $\delta[e(t) - s]$, explain why it should be accompanied by the Faddeev-Popov (FP) determinant $\Delta_{FP} = \det[\partial_t \delta(t - t')]$. Explain *briefly* why inclusion of this determinant is equivalent to an addition to the classical action of the "Faddeev-Popov ghost" action

$$I_{FP} = \int dt\, \{\, ib\dot{c}\}\ ,$$

where $b(t)$ and $c(t)$ are *anticommuting* variables.

(b) A mechanical system is described by the action

$$I[q,p;\lambda] = \int dt\, \{\dot{q}^I p_I - \lambda^i \varphi_i(q,p)\}\ , \quad (I = 1, \ldots, N; \quad i = 1, \ldots, n < N)\,,$$

where the constraint functions satisfy $\{\varphi_i, \varphi_j\}_{PB} = f_{ij}{}^k \varphi_k$ for *constants* $f_{ij}{}^k$. Why do these constants satisfy the identity $f_{[ij}{}^l f_{k]l}{}^m \equiv 0$?

Write down the gauge transformation of the canonical variables generated by $\epsilon^i \varphi_i$ for infinitesimal parameters $\epsilon^i(t)$. Given that invariance of the action also requires

$$\delta\lambda^i = \dot{\epsilon}^i + \epsilon^j \lambda^k f_{jk}{}^i\ ,$$

and that the gauge invariance is fixed by imposing $\lambda^i(t) = \bar{\lambda}^i$, where $\bar{\lambda}^i$ are constants, find the FP ghost action.

(c) Write down the FP ghost action for the closed Nambu-Goto string. What are the conformal dimensions of the FP worldsheet ghost fields? Explain briefly how consideration of conformal invariance, now including FP ghosts, restricts the spacetime dimension D to the critical dimension, $D = 26$.

The central charge of the Virasoro algebra associated to a generic 'bc system' for which b has conformal dimension J is $c = -2(6J^2 - 6J + 1)$. Use this fact to explain *briefly* why the worldsheet fermion fields of the 'spinning' string (either Ramond or Neveu-Schwarz) contribute $D/2$ to the total central charge, and how this leads to the critical dimension $D = 10$ after account is taken of additional *commuting* FP ghost fields arising from additional gauge invariances of the spinning string.

4 Write an essay explaining (*without detailed computation*) why string theory is a perturbative theory of quantum gravity. You may use units for which $8\pi T = 1$ for string tension T, and you should include the following topics:

- The presence of a massless spin-2 particle in the closed string spectrum.
- The relation of particles in the string spectrum to vertex operators, and the use of vertex operators (in the path integral formulation) for the computation of scattering amplitudes.
- The "dual resonance" property of string amplitudes that leads to the conclusion that the spin-2 graviton is exchanged in the scattering of string states. You may wish to illustrate your discussion in terms of the Virasoro amplitude

$$A(s,t) \propto \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(-1-u)}{\Gamma(u+2)\Gamma(s+2)\Gamma(t+2)} \qquad (u = -4 - s - t).$$

- Some features of the string-loop expansion, and the ultra-violet finiteness of the one string-loop contribution to the vacuum energy.

END OF PAPER