UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 1:30 pm to 4:30 pm

PAPER 305

THE STANDARD MODEL

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Given that the Dirac field $\psi(x)$ satisfies the Dirac equation for a massive particle, show that $\psi^c(x) \equiv \hat{C}\psi(x)\hat{C}^{-1} = C\bar{\psi}^T(x)$ satisfies the Dirac equation, where $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$, $C^{-1} = C^\dagger$, and $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Find an expression for $\hat{C}\bar{\psi}(x)\hat{C}^{-1}$ in terms of $\psi^T(x)$ and C^{-1}.

(b) Recall the expansion of the complex vector field $V^\mu(x)$,

$$V^\mu(x) = \sum_{p,\lambda} \left[\varepsilon^{\mu,\lambda}(p)\, a^\lambda(p)\, e^{-ip\cdot x} + \varepsilon^{\mu,\lambda *}(p)\, c^{\lambda\dagger}(p)\, e^{ip\cdot x}\right] .$$

Briefly explain the meaning of $a^\lambda(p)$, $c^{\lambda\dagger}(p)$ and $\varepsilon^{\mu,\lambda}(p)$. Determine how $V^\mu(x)$ transforms under charge-conjugation, $\hat{C}V^\mu(x)\hat{C}^{-1}$. [*You may assume that*

$$\hat{C}a^\lambda(p)\hat{C}^{-1} = \eta_C\, c^\lambda(p)\,, \qquad \hat{C}c^{\lambda\dagger}(p)\hat{C}^{-1} = \eta_C\, a^{\lambda\dagger}(p)\,,$$

where η_C is a complex phase.]

(c) Derive expressions for the transformation of $\bar{\psi}(x)\gamma^\mu\psi(x)$ and $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ under charge-conjugation. [*Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.*]

(d) What restriction is placed on η_C if $V^\mu(x)$ is a real field? Given that the interaction between a photon and an electron is invariant under charge-conjugation, deduce the value of η_C for the photon field $A^\mu(x)$.

(e) How does $\bar{\psi}(x)\gamma^\mu(g_v - g_a\gamma^5)\psi(x)V_\mu$, where g_v and g_a are real constants, transform under CP (a parity transformation followed by a charge-conjugation transformation)? [*You may assume that*

$$\hat{P}\psi(x)\hat{P}^{-1} = \eta_P\,\gamma^0\psi(x_P)\,, \quad \hat{P}\bar{\psi}(x)\hat{P}^{-1} = \eta_P^*\,\bar{\psi}(x_P)\gamma^0\,, \quad \hat{P}V^\mu(x)\hat{P}^{-1} = \mathbb{P}^\mu_{\ \nu}\,V^\nu(x_P)\,,$$

where η_P is a complex phase, $x_P^\mu = (x^0, -\mathbf{x})$, and $\mathbb{P}^\mu_{\ \nu} = \mathrm{diag}(1,-1,-1,-1)$.]

(f) What does the result in (e) imply for the interactions of the Z boson in the Standard Model? Explain whether or not the same argument holds for the interactions of the W bosons. [*You may assume that the Z and W boson fields have the same η_C as the photon field.*]

2 An $SU(N)$ gauge theory involving a real scalar field ϕ_a in the adjoint representation and a gauge field B^a_μ $(a = 1, 2, \dots N^2 - 1)$ has Lagrangian,

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a - V(\phi_a\phi_a) - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}\,,$$

where $D_\mu = \partial_\mu + ig\,t^a B^a_\mu$, $(t^a)_{bc} = -if^{abc}$ and $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - gf^{abc}B^b_\mu B^c_\nu$. The generators t^a satisfy $[t^a, t^b] = if^{abc}\,t^c$ where f^{abc} are the (antisymmetric) structure constants.

(a) Consider an $SU(2)$ gauge theory ($f^{abc} = \epsilon^{abc}$) with potential

$$V(\phi_a\phi_a) = \frac{1}{2}m^2\phi_a\phi_a + \frac{\lambda}{4}(\phi_a\phi_a)^2\,, \quad \lambda > 0, \quad m^2 < 0\,.$$

Why, without loss of generality, can we take the vacuum to be $(0, 0, v)^T$ and the fluctuations of ϕ about the vacuum to be $\phi(x) = (0, 0, v + \eta(x))^T$, where v and $\eta(x)$ are real? Discuss how the symmetry is spontaneously broken by the vacuum, identify the unbroken symmetry and write the Lagrangian in terms of physical fields. Give the masses of the physical fields (ignoring any quantum corrections) and briefly summarize their interactions.

Explain briefly in what ways this theory, after adding couplings to fermions, could be a suitable description of weak and electromagnetic interactions. In what crucial respects does it differ from the electroweak part of the Standard Model?

(b) Now consider an $SU(3)$ gauge theory and suppose that $\Phi = \phi_a t^a$ acquires a vacuum expectation value

$$\langle\Phi\rangle = \Phi_0 = v\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

The gauge bosons acquire masses from the term $\frac{1}{2}g^2(f^{abc}B^b_\mu\phi_c)(f^{aef}B^{\mu e}\phi_f)$. Show that this can be written as

$$-g^2\,\mathrm{Tr}\left([t^a, \Phi][t^b, \Phi]\right)B^a_\mu B^{b\mu}\,.$$

Hence, ignoring any quantum corrections and considering this term with $\Phi = \Phi_0$, show that 4 gauge bosons remain massless. Assuming that the other 4 gauge bosons all acquire the same mass, find this mass in terms of v and g. What is the symmetry which remains after spontaneous symmetry breaking?

[*Hint:* $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ *and* $t^a = \frac{1}{2}\lambda^a$ *where*

$$\lambda^1 = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda^2 = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda^3 = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda^4 = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} \quad \lambda^6 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \quad \lambda^7 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.]$$

3 (a) Consider the neutral pseudoscalar charm mesons D^0 and $\bar{D}^0$ with quark flavour content $\bar{u}c$ and $\bar{c}u$ respectively. These can mix in an analogous way to K^0 and $\bar{K}^0$ mesons. Draw a Feynman diagram representing one of the leading-order Standard Model contributions to D^0 – $\bar{D}^0$ mixing.

(b) Let H' represent the relevant weak Hamiltonian and denote the matrix elements of H' by

$$\begin{pmatrix} \langle D^0|H'|D^0\rangle & \langle D^0|H'|\bar{D}^0\rangle \\ \langle \bar{D}^0|H'|D^0\rangle & \langle \bar{D}^0|H'|\bar{D}^0\rangle \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} .$$

Derive a relationship between the matrix elements assuming invariance under CPT. What condition would relate the matrix elements R_{12} and R_{21} if the weak interaction were invariant under CP? [*Hints: Consider the rest frame and assume that we can choose conventions such that* $\hat{C}\hat{P}\,|D^0\rangle = -\,|\bar{D}^0\rangle$ *and* $\hat{C}\hat{P}\,|\bar{D}^0\rangle = -\,|D^0\rangle$.]

(c) Draw a tree-level Feynman diagram for each of the processes $\bar{D}^0 \to K^+\pi^-$ and $\bar{D}^0 \to K^-\pi^+$ and comment on their expected relative rates.

(d) Neglecting D^0 – $\bar{D}^0$ mixing, consider the decay $\bar{D}^0(p) \to K^+(k)\,\pi^-(q)$ for which the relevant part of the effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{cs} V_{ud}^* \big[\bar{c}\gamma^\alpha(1-\gamma^5)s\big]\big[\bar{d}\gamma_\alpha(1-\gamma^5)u\big] .$$

Explain why the following two equalities hold

$$\big\langle K^+(k)|\bar{c}\gamma^\mu(1-\gamma^5)s|\bar{D}^0(p)\big\rangle = \big\langle K^+(k)|\bar{c}\gamma^\mu s|\bar{D}^0(p)\big\rangle = (p+k)^\mu f_+(q^2) + (p-k)^\mu f_-(q^2) ,$$

where $q = p - k$ and f_+ and f_- are appropriate Lorentz scalar functions.

Hence, using the approximation $m_\pi = 0$, show that the tree-level decay rate for this process can be written as

$$\Gamma = A\, G_F^2 \left| V_{ud}^* V_{cs}\, F_\pi\, f_+(0) \right|^2$$

where F_π is the decay constant of the π^- and A is a constant which you should write in terms of m_D and m_K (the D and K masses). [*Hints:* F_π *is defined by* $\big\langle \pi^-(q)|\bar{d}\gamma^\mu\gamma^5 u|0\big\rangle = -i\sqrt{2}\,F_\pi\, q^\mu$. *The decay rate for* $A(p) \to B(k) + C(q)$ *is,*

$$\Gamma(A \to BC) = \frac{1}{2m_A} \int \frac{d^3k}{(2\pi)^3 2k^0} \int \frac{d^3q}{(2\pi)^3 2q^0}\, (2\pi)^4 \delta^{(4)}(p-k-q)\, |\mathcal{M}|^2 ,$$

where m_A *is the mass of particle* A.]

4 (a) Consider the deep inelastic scattering of an electron off a hadron H of rest mass M via a virtual photon, $e(p)H(P_H) \to e(p')X(P_X)$. Draw a Feynman diagram for the process. Treating electrons as massless and working in the rest frame of H, show that,

$$\frac{d\sigma}{d^3p'} = \frac{e^4}{8(2\pi)^2 MEE'\,q^4} L_{\mu\nu}(p,p')\,W_H^{\mu\nu}(q,P_H)$$

where E and E' are the initial and final electron energies, $q = p - p'$ and

$$W_H^{\mu\nu}(q,P_H) = \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(q + P_H - P_X)\,\langle H(P_H)|J^\mu|X(P_X)\rangle\,\langle X(P_X)|J^\nu|H(P_H)\rangle\ ,$$

where the sum over X implicitly includes the appropriate integral over P_X and spin summing/averaging, J^μ is the relevant electromagnetic current, and you should give an expression for $L_{\mu\nu}$ in terms of p and p'. [*Hints: The following expression may be used without proof:* $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, *and the differential cross section for* $A(p_A) + B(p_B) \to C(p_C) + D(p_D)$ *is*

$$d\sigma = \frac{1}{|\vec{v}_A - \vec{v}_B|}\frac{1}{4p_A^0 p_B^0}\left(\frac{d^3p_C}{(2\pi)^3 2p_C^0}\right)\left(\frac{d^3p_D}{(2\pi)^3 2p_D^0}\right)(2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)\,|\mathcal{M}|^2\,.]$$

(b) Explain why we can write

$$W_H^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)F_1(x,q^2) + \left(P_H^\mu - \frac{P_H\cdot q}{q^2}q^\mu\right)\left(P_H^\nu - \frac{P_H\cdot q}{q^2}q^\nu\right)\frac{F_2(x,q^2)}{\nu}\,,$$

where F_1 and F_2 are Lorentz scalar functions, $\nu = P_H\cdot q$ and $x = -q^2/(2\nu)$.

(c) In certain cases, $W_H^{\mu\nu}$ can be approximated by considering incoherent elastic scattering from "partons" with momentum $k = \xi P_H$ where ξ is the momentum fraction of the initial hadron's momentum,

$$W_H^{\mu\nu}(q,P_H) = \int_0^1 d\xi \sum_f Q_f^2\,\tilde{W}^{\mu\nu}(\xi,q,P_H)\big[q_f(\xi) + \bar{q}_f(\xi)\big]$$

where $q_f(\xi)$, $\bar{q}_f(\xi)$ is the probability distribution for a quark, antiquark of flavour f and charge Q_f, and

$$Q_f^2\,\tilde{W}^{\mu\nu}(\xi,q,P_H) =$$
$$\frac{1}{4\pi\xi}\int\frac{d^3k'}{(2\pi)^3 2E_{k'}}(2\pi)^4\delta^{(4)}(q + \xi P_H - k')\frac{1}{2}\sum_{\text{spins}}\langle q_f(\xi P_H)|J^\nu|q_f(k')\rangle\,\langle q_f(k')|J^\mu|q_f(\xi P_H)\rangle\ .$$

(i) Neglecting quark and hadron masses, show that the only terms in $\tilde{W}^{\mu\nu}(k,q)$ which contribute are

$$\tilde{W}^{\mu\nu}(\xi,q,P_H) = \frac{1}{E_{k'}}\delta(E_q + \xi E_{P_H} - E_{k'})\left[\xi P_H^\mu P_H^\nu - \frac{P_H\cdot q}{2}\,g^{\mu\nu}\right]\,.$$

[*Hint: Show that* $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$.]

(ii) Finally, show that $2xF_1(x,q^2) = F_2(x,q^2)$. [*Hint: Show that* $\delta(E_q + \xi E_{P_H} - E_{k'})/E_{k'} = 2\delta(E_{k'}^2 - (\xi E_{P_H} + E_q)^2) = \delta(x - \xi)/\nu$.]

END OF PAPER