UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Briefly explain how Feynman diagrams may be used to evaluate the integral

$$\mathcal{Z}(m,\lambda) = \int_{\mathbb{R}} \mathrm{d}\phi \, \exp\left(-\frac{m^2}{2}\phi^2 - \frac{\lambda}{6!}\phi^6\right)$$

as an asymptotic series in λ. Your answer should include an account of Wick's theorem, propagators, vertices and symmetry factors from the point of view of path integrals.

Draw all Feynman diagrams with at most two vertices that contribute to $\mathcal{Z}(m,\lambda)/\mathcal{Z}(m,0)$. Hence compute this ratio, correct to λ^2 accuracy.

2 Consider the action

$$S[\phi] = \int \left(\frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + \frac{\lambda}{n!}\phi^n\right) \mathrm{d}^d x$$

for a single scalar field with a monomial interaction in d Euclidean dimensions. Give all values of (n,d) for which

i.) there is non-trivial wavefunction renormalization at 1-loop,

ii.) the coupling λ is classically relevant or marginal,

iii.) no new types of vertices are generated in the continuum quantum effective action.

[Consider each case separately.]

Now consider a real scalar field in $d=4$ with mass m and a quartic interaction λ. Using a cut-off at momentum Λ_0, compute the quartic vertex in the quantum effective action to 1-loop accuracy. Show that it is finite as $\Lambda_0 \to \infty$. What other effective interactions are generated by the Feynman diagrams you have considered? What happens to these as $\Lambda_0 \to \infty$?

3 Consider the action

$$S[A,\phi] = \int \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \nabla^\mu \bar{\phi}\, \nabla_\mu \phi + m^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4 \right) \mathrm{d}^4 x$$

where A_μ is a U(1) gauge field, ϕ is a complex scalar and $\nabla_\mu \phi = \partial_\mu \phi - \mathrm{i} A_\mu \phi$. Obtain the Noether current $j_\mu(x)$ associated to the charge transformations

$$\phi \mapsto \mathrm{e}^{\mathrm{i}\alpha} \phi\,, \qquad \bar{\phi} \mapsto \mathrm{e}^{-\mathrm{i}\alpha} \bar{\phi}\,, \qquad A_\mu \mapsto A_\mu\,.$$

A Ward–Takahashi identity associated with this current states that

$$\partial^\mu \langle j_\mu(x) \phi(y) \bar{\phi}(z) \rangle = -\mathrm{i}\delta^4(x-y)\, \langle \phi(y) \bar{\phi}(z) \rangle + \mathrm{i}\delta^4(x-z) \langle \phi(y) \bar{\phi}(z) \rangle\,.$$

Show that

$$(p-p')_\mu \Gamma^\mu(p,p') = \mathrm{i} \left(\Delta(p')^{-1} - \Delta(p)^{-1} \right)\,,$$

where $\Gamma^\mu(p,p')$ is the sum of all 1PI diagrams with one external ϕ of momentum p, one external $\bar{\phi}$ of momentum p' and one external A_μ, and $\Delta(p)$ is the exact scalar propagator in momentum space.

Construct a further Ward–Takahashi identity to show that

$$k_\mu \Gamma^{\mu\nu}(k,p,p') = \mathrm{i}\Gamma^\nu(p, k-p') - \mathrm{i}\Gamma^\nu(p+k, p')\,,$$

where $\Gamma^{\mu\nu}(k,p,p')$ is the sum of all 1PI diagrams with two external gauge fields, one of momentum k, and two external scalars as before.

4 Consider a non–Abelian gauge theory in four dimensions, minimally coupled to a massless scalar field ϕ in the adjoint representation. BRST transformations act as

$$Q\phi = [c,\phi]$$

on the scalar and

$$Qc = -\frac{1}{2}[c,c]$$

on the ghost c. Show that $Q^2\phi = 0$.

Suppose the path integral is defined using the gauge-fixing functional

$$f^{\mathrm{a}}(A,\phi,h) = \partial^\mu A_\mu^{\mathrm{a}} + \frac{1}{\sqrt{2}} n^\mu \partial_\mu \phi^{\mathrm{a}} - \frac{\mathrm{i}}{2} h^{\mathrm{a}}\,,$$

where h^{a} is the Nakanishi-Lautrup field and n^μ is a fixed, constant vector. Show that the propagator for the scalar field is independent of the momentum in the n^μ direction. Find the Feynman rules for ghost vertices in momentum space in this gauge.

END OF PAPER