UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 1:30 pm to 3:30 pm

PAPER 303

STATISTICAL FIELD THEORY

Attempt Question 1 and either Question 2 or 3.

There are ***THREE*** *questions in total.*

Question 1 carries 40 marks.
Questions 2 and 3 each carry 60 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A system in d spatial dimensions is described by a real, local order parameter $\phi(\mathbf{x})$.

What is the most general form of the free energy in the case that the system has rotational invariance and a $\mathbf{Z}_2$ symmetry $\phi(\mathbf{x}) \mapsto -\phi(\mathbf{x})$?

Within mean field theory, explain how the free energy can be used to describe the phase transition as the temperature passes through the critical temperature $T = T_c$. How does the situation change if the $\mathbf{Z}_2$ symmetry is relaxed?

Consider the free energy of the form

$$F[\phi] = \int d^d x \, \left\{ \frac{\gamma}{2}(\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2 + \lambda_n \phi^{2n} \right\}.$$

Determine the mean field critical exponent $\langle\phi\rangle \sim (T_c - T)^\beta$ in the ordered phase. Determine the mean field critical exponent $c \sim (T_c - T)^{-\alpha}$ in the ordered phase, where c is the heat capacity.

Around the Gaussian fixed point, fluctuations give a contribution to the heat capacity with $\alpha = 2 - d/2$. Show that these are the dominant contribution in dimension $d < d_c$ for some d_c that you should determine.

2

In $d = 3$ dimensions, the free energy for a real order parameter $\phi(\mathbf{x})$ is given by

$$F[\phi] = \int d^3 x \, \left\{ \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{\mu^2}{2}\phi^2 + g\phi^4 + \lambda\phi^6 \right\}.$$

What is the dimension of ϕ at the Gaussian fixed point? Using this naive dimensional analysis, is ϕ^2 relevant, irrelevant or marginal? What about ϕ^4 and ϕ^6?

State the three steps of the renormalisation group. Start with $g = 0$ and $\mu^2 = \mu_0^2$ and $\lambda = \lambda_0$. To leading order in λ_0, determine how the coupling g flows under the renormalisation group. (You may leave your answer in integral form.)

To order λ_0^2, explain the steps necessary to determine the beta function for λ. You need not compute the magnitude of the overall numerical coefficient of the beta function, but should explain which term in the expansion of the free energy will contribute, and the relevance of Wick's theorem in evaluating this term. Re-evaluate whether the ϕ^6 interaction is relevant, irrelevant or marginal.

3

A system in d spatial dimensions is described by two local order parameters, $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$. The free energy is invariant under a $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry, in which $\phi_1 \mapsto -\phi_1$ and, independently, $\phi_2 \mapsto -\phi_2$. The leading terms are

$$F[\phi_1, \phi_2] = \int d^d x \, \left\{ \sum_{i=1}^{2} \left[\frac{1}{2} \nabla \phi_i \cdot \nabla \phi_i + \frac{\mu_i^2}{2} \phi_i^2 + g_i \phi_i^4 \right] + \lambda \phi_1^2 \phi_2^2 \right\}.$$

The question parts a), b) and c) below are to be considered independently; the parameters μ_i^2, g_i and λ take different values in each.

a) Suppose that $g_1 = g_2 = \lambda/2 = g$. Assuming mean field theory, describe the phase diagram in the (μ_1^2, μ_2^2) plane. Identify the points or lines on the (μ_1^2, μ_2^2) plane where phase transitions occur, and describe the nature of the transition. (You need not compute critical exponents.)

b) Suppose that $\mu_i^2 = 0$ for $i = 1, 2$. In $d = 4 - \epsilon$ dimensions, the beta functions are given by

$$\begin{aligned} \frac{dg_1}{ds} &= \alpha_1 g_1 - (36 g_1^2 + \lambda^2) A \\ \frac{dg_2}{ds} &= \alpha_2 g_2 - (36 g_2^2 + \lambda^2) A \\ \frac{d\lambda}{ds} &= \alpha_3 \lambda - (8\lambda^2 + 12 \lambda g_1 + 12 \lambda g_2) A \end{aligned}$$

for some constants α_1, α_2, α_3 and A.

What are the values of α_1, α_2 and α_3? Find the fixed points. What is the symmetry of the theory at each fixed point?

c) Suppose that $\mu_1^2 = \mu_2^2 = \mu^2$ and $g_1 = g_2 = \lambda/2 = g$. Briefly explain how one can distinguish between the two different phases in $d = 2$ dimensions.

END OF PAPER