UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2018 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a matrix Lie group with matrix Lie algebra $\mathbb{L}(G)$. State the definition of the exponential map from $\mathbb{L}(G)$ to G. Show that, for each $X \in \mathbb{L}(G)$, the set

$$G_X = \{\mathrm{Exp}(tX); t \in I \subseteq \mathbb{R}\}$$

is a Lie subgroup of G for a suitable choice of an interval I on the real line. Show that, up to isomorphism, there are two distinct possibilities for the Lie group G_X.

Now let $G = SL(2,\mathbb{R})$, the group of 2×2 matrices with real entries having determinant equal to one. Find the corresponding Lie algebra $\mathbb{L}(G)$. In particular, find basis elements L_0, L_+ and $L_- \in \mathbb{L}(G)$ obeying,

$$L_0^2 = \mathbb{I}_2 \qquad\qquad L_+^2 = L_-^2 = 0$$

where $\mathbb{I}_2$ denotes the 2×2 unit matrix. Evaluate the structure constants of the Lie algebra in this basis.

Identify the Lie subgroups $G_X \subset G$, for $X = L_0$, $X = L_+$ and $X = L_+ - L_-$ giving the corresponding interval $I \subseteq \mathbb{R}$ in each case. Writing a general element of the Lie algebra as $X = \alpha_0 L_0 + \alpha_+ L_+ + \alpha_- L_-$ for some α_0, $\alpha_\pm \in \mathbb{R}$, find a necessary and sufficient condition on the coefficients α_0, $\alpha_\pm$ such that G_X is compact. [*Hint: You may use the fact that, provided its eigenvalues are distinct, the matrix $X \in \mathbb{L}(G)$ can be diagonalised by a similarity transformation: $X = P\Lambda P^{-1}$ for some non-singular matrix P where Λ is diagonal. In the degenerate case, where the eigenvalues are not distinct, you may assume instead that the matrix is nilpotent: $X^2 = 0$.*]

2 Write an essay on the *Killing form* of a finite-dimensional Lie algebra. In addition to a discussion of the general properties of the Killing form, your essay should include a proof that a non-degenerate Killing form can only arise for a semi-simple Lie algebra. In the case of a simple, complex Lie algebra $\mathfrak{g}$ you should also show the Killing form is non-degenerate when restricted to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Finally you should state without proof the Euclidean property enjoyed by the Killing form on a particular real subspace of $\mathfrak{h}$ and explain the significance of this for the root system of $\mathfrak{g}$.

3 *In this question you may use without proof any general results from the theory of simple, complex, finite-dimensional Lie algebras and their representations provided they are clearly stated.*

Starting from the general constraints obeyed by the Cartan matrix of a simple, complex, finite dimensional Lie algebra, classify all such Lie algebras $\mathfrak{g}$ of rank two. In particular give the Cartan matrix for each rank two Lie algebra and draw the corresponding Dynkin diagram. In each case, find the ratio of lengths of the two simple roots and the angle between them. [*In labeling the entries of the Cartan matrix you should adopt the convention that, whenever the two simple roots have unequal length, the first row/column corresponds to the longer of the two.*]

By considering all root strings of the form $\beta + n\alpha$ where α is a simple root, β is a root and n is an integer, find all roots of each of these rank two Lie algebras. In each case you should give the roots as linear combinations of the simple roots. Hence deduce the dimension of each Lie algebra.

Consider the $\mathbb{L}_{\mathbb{C}}(SU(2))$ subalgebra of $\mathfrak{g}$, denoted $sl(2)_\alpha$, associated with each simple root α of $\mathfrak{g}$. For each rank two Lie algebra $\mathfrak{g}$ and each simple root α, identify the adjoint representation of $\mathfrak{g}$ as a representation of the subgroup $sl(2)_\alpha$. [*Recall that all finite-dimensional representations of* $\mathbb{L}_{\mathbb{C}}(SU(2))$ *can be decomposed as* $R(\Lambda_1) \oplus R(\Lambda_2) \oplus \ldots \oplus R(\Lambda_L)$ *for some* L*, where* $R(\Lambda)$ *is the irreducible* $\mathbb{L}_{\mathbb{C}}(SU(2))$ *representation of highest weight* $\Lambda \in \mathbb{Z}_{\geqslant 0}$*. Your answer should specify the highest weights* Λ *appearing in this decomposition in each case.*]

4 *In this question you may use without proof the invariance property of the Killing form on a Lie algebra.*

Consider a non-abelian gauge theory with gauge group G and gauge field $A^\mu(x)$ taking values in the Lie algebra $\mathbb{L}(G)$. The theory also contains a scalar field $\phi(x)$ taking values in the representation space V of a unitary representation R of $\mathbb{L}(G)$. Give the definition of the covariant derivative $D_\mu\phi$ for the scalar field and derive its transformation property under the infinitesimal gauge transformation $A_\mu \to A_\mu + \delta_X A_\mu$, $\phi \to \phi + \delta_X \phi$ where,

$$\begin{aligned} \delta_X A^\mu &= -\epsilon\partial^\mu X + \epsilon\,[X, A^\mu] \qquad (*) \\ \delta_X \phi &= \epsilon R(X)\phi. \end{aligned}$$

Here $\epsilon \ll 1$ is a small parameter and X is a (space-time dependent) element of the Lie algebra $\mathbb{L}(G)$.

Define the field-strength tensor $F^{\mu\nu}$ for the gauge field A^μ and state (without proof) its transformation property under the gauge transformation $(*)$. Hence write down a consistent Lorentz invariant Lagrangian density $\mathcal{L}$ for the theory in question and show that it is gauge invariant.

Now consider the case where R is the adjoint representation so that the representation space V can be identified with the Lie algebra $\mathbb{L}(G)$ itself and the inner product on V can be identified with the Killing form on $\mathbb{L}(G)$. Let $\{T^a; a = 1, \ldots, \mathrm{dim}G\}$ be a basis for $\mathbb{L}(G)$. Let $A^\mu = A^\mu_a T^a$, $\phi = \phi_a T^a$, where summation over the index a is implied, be the corresponding expansions of the fields. Write out the Lagrangian density $\mathcal{L}$ explicitly in terms of the components A^μ_a and ϕ_a of the fields and those of their (ordinary) derivatives, $\partial_\nu A^\mu_a$ and $\partial_\nu \phi_a$, as well as the structure constants $f^{ab}_{\ \ c}$ of $\mathbb{L}(G)$ and components $\kappa^{ab} = \kappa(T^a, T^b)$ of the Killing form in the chosen basis.

END OF PAPER