UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2018 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question, you should neglect fermion masses.

(i) Write down the Clifford algebra of γ^μ matrices.

(ii) Consider the process $e^-(p,s)e^+(q,r) \to \mu^-(p',s')\mu^+(q',r')$ at the lowest non-trivial order in QED where p,q,p',q' are 4-momenta and r,s,r',s' are spins. Write down the photon propagator Feynman rule in Feynman gauge. Write down the Feynman rule for the interaction between an electron and a photon.

(iii) Represent the modulus squared of the amplitude in Feynman diagram format.

(iv) Calculate $|\bar{A}|^2$, the amplitude squared summed over final state spins and averaged over initial state spins in terms of only e and the centre of mass frame scattering angle θ between e^- and μ^-. You should derive any relations you use for the trace of products of gamma matrices from the Clifford algebra.

(v) The centre of mass frame differential cross-section for the process is given by

$$\frac{d\sigma}{d\Omega} = \frac{|\bar{A}|^2}{64\pi^2 s},$$

where $s = (p+q)^2$ and Ω is the solid scattering angle. Calculate the total cross-section in terms of $\alpha = e^2/(4\pi)$, showing that it is of the form:

$$\sigma = B\alpha^j s^k t^l u^m,$$

where $t = (p-p')^2, u = (p-q')^2$. You should determine the numerical constants B, j, k, l and m in the process.

2

(i) Write down an equation relating the time independent Hamiltonian H, time t, an operator in the Schrödinger picture O_S, and the corresponding operator in the Heisenberg picture $O_H(t)$ and thus calculate $[H, O_H]$ in terms of O_H.

(ii) Take a real quantum field in the Heisenberg picture $\phi_1(x^\mu)$ of mass m_1 and its conjugate momentum $\pi_1(x^\mu)$. Write down the Lagrangian density for the free theory.

(iii) Calculate the Hamiltonian for the *free* quantum field theory of ϕ_1 in terms of ϕ_1, its conjugate momentum π_1 and $\underline{\nabla}\phi_1$.

(iv) Write down the commutators $[\phi_1(t, \underline{x}),\ \phi_1(t, \underline{y})]$, $[\pi_1(t, \underline{x}),\ \pi_1(t, \underline{y})]$ and $[\phi_1(t, \underline{x}),\ \pi_1(t, \underline{y})]$ where $\underline{x}$ and $\underline{y}$ are position 3-vectors.

(v) Using the information you have written down thus far, show that $\phi_1(x^\mu)$ satisfies the Klein-Gordon equation.

(vi) Add to the Lagrangian a free field ϕ_2 of mass m_2. Suppose that the transformation

$$\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta, \qquad \phi_2 \to \phi_2 \cos\theta - \phi_1 \sin\theta, \qquad \theta \text{ constant}$$

leaves the Lagrangian density invariant. Using Noether's theorem, compute the Noether current and the associated conserved charge Q.

(vii) Now explicitly compute dQ/dt and hence derive a relation between m_1 and m_2.

3

Consider the Lagrangian density for a real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{3!}\phi^3.$$

(i) Draw the lowest order Feynman diagrams that contribute to the amplitude for $\phi(p)\phi(q) \to \phi(p')\phi(q')$ scattering, where p, q, p', q' represent 4-momenta.

(ii) Next, calculate the modulus squared of the scattering amplitude $|A|^2$ as a function of $s = (p+q)^2$, $t = (p-p')^2$, λ and m. You need not derive the Feynman rules.

(iii) The total cross-section may be written

$$\sigma = \int \frac{d^3p'}{(2\pi)^3\ 2E_{p'}} \frac{d^3q'}{(2\pi)^3\ 2E_{q'}} \frac{|A|^2}{2\kappa^{1/2}(s, m^2, m^2)} (2\pi)^4 \delta^4(q' - p - q + p'),$$

where $\kappa(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Show that

$$\sigma = X \int_{t_{min}}^{t_{max}} dt\ \frac{|A|^2}{s(s - Ym^2)},$$

determining the constants X and Y. Determine t_{min} and t_{max} in terms of s and m.

[Hint: work in the centre of mass frame]

(iv) Check that both sides of your final expression have the same mass dimension.

4

Maxwell's Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu = A_\mu(x)$ is the 4-vector potential.

(i) Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi,$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.

(ii) A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is defined such that $\eta_{\mu\nu} x^\mu x^\nu$ is Lorentz invariant. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \, .$$

(iii) Use this result to show that an infinitesimal Lorentz transformation of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \alpha \omega^\mu{}_\nu$$

implies that $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (α is infinitesimal).

(iv) Working to first order in α show that, under an active infinitesimal Lorentz transformation, the Lagrangian density transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) - \alpha \omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}(x)$$

and hence that the variation of the Lagrangian density $\delta\mathcal{L}$ is a total derivative

$$\delta\mathcal{L} = -\partial_\mu (\alpha \omega^\mu{}_\nu x^\nu \mathcal{L}).$$

(v) Use Noether's theorem and the spacetime translational invariance of the action (i.e. $x^\mu \to x^\mu - \beta\epsilon^\mu$, where β is an infinitesimal constant and ϵ^μ is a constant 4-vector), to construct an energy momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant.

(vi) Consider a new tensor given by

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu} \partial_\rho A^\nu.$$

Show that this object defines four conserved currents (you may find the equations of motion useful). Moreover, show that it is symmetric, gauge invariant and traceless.

END OF PAPER