UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 1:30 pm to 3:30 pm

PAPER 214

PERCOLATION AND RANDOM WALKS ON GRAPHS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the FKG inequality for bond percolation on $\mathbb{Z}^d$. Let $A_1, \ldots, A_n$ be increasing events for bond percolation in $\mathbb{Z}^d$ having the same probability. Prove that

$$\mathbb{P}_p(A_1) \geqslant 1 - (1 - \mathbb{P}_p(A_1 \cup \ldots \cup A_n))^{1/n}.$$

(b) Let $d \geqslant 2$ and consider bond percolation on $\mathbb{Z}^d$ with $p > p_c$. For each m denote $B(m) = [-m, m]^d \cap \mathbb{Z}^d$.

(i) Prove that uniformly in $n \in \mathbb{N}$ we have

$$\mathbb{P}_p(B(m) \longleftrightarrow \partial B(n+m)) \to 1 \quad \text{as } m \to \infty.$$

(ii) Let $F_L(n) = \{x \in B(n) : x_1 = -n\}$ and $F_R(n) = \{x \in B(n) : x_1 = n\}$ be the "left" and "right" faces respectively of $B(n)$. Show that for $i \in \{L, R\}$ uniformly in $n \in \mathbb{N}$ we have

$$\mathbb{P}_p(B(m) \longleftrightarrow F_i(n+m)) \to 1 \quad \text{as } m \to \infty.$$

(iii) Let $\mathrm{LR}(n)$ denote the event that there exists a left to right crossing of $B(n)$ which lies in $B(n)$, i.e. a path of open edges of $B(n)$ connecting $F_L(n)$ to $F_R(n)$. Prove that

$$\mathbb{P}_p(\mathrm{LR}(n)) \to 1 \quad \text{as } n \to \infty.$$

2

Consider bond percolation on $\mathbb{Z}^2$ with parameter $p \in (1/2, 1)$ and let $B(m) = [-m, m]^2 \cap \mathbb{Z}^2$. Let C denote the cluster of the origin and let $\theta(p) = \mathbb{P}_p(|C| = \infty)$.

(a) Let $R(m)$ denote the number of vertices in $B(m)$ that belong to the a.s. unique infinite cluster. Show that

$$\mathbb{P}_p\left(R(m) \geqslant \frac{\theta(p)|B(m)|}{2}\right) \geqslant \frac{\theta(p)}{2}.$$

(b) Show that there exists a positive constant c_1 so that for all $n \in \mathbb{N}$

$$\mathbb{P}_p(n \leqslant |C| < \infty) \geqslant \exp(-c_1\sqrt{n}).$$

(Hint: Consider the set of vertices in $B(m)$ that are connected to $\partial B(m)$ by open paths of edges.)

(c) Using duality or otherwise prove that there exists a positive constant c_2 so that for all $n \in \mathbb{N}$

$$\mathbb{P}_p(|C| = n) \leqslant \exp(-c_2\sqrt{n}).$$

(You may assume that if G is a connected subgraph of $\mathbb{Z}^2$ on n vertices, then there exists a dual circuit containing G in its interior with the property that each of its edges crosses an edge on the edge boundary of G. Moreover, it has size at least $\alpha\sqrt{n}$, where α is a positive constant. Properties of subcritical percolation may be used without proofs provided they are stated clearly.)

3

Let $G = (V, E)$ be a finite connected graph on n vertices with unit resistances on the edges. For any two vertices a and b let $d_G(a, b)$ stand for the number of edges on the shortest path in G joining a and b.

(a) State Thomson's and Rayleigh's monotonicity principles for effective resistance. Show that for all vertices a, b

$$R_{\text{eff}}(a, b) \leqslant d_G(a, b).$$

(b) Let X be a simple random walk on G starting from a. The cover time is the first time X has visited all the vertices of G at least once, i.e.

$$\tau_{\text{cov}} = \min\{t \geqslant 0 : \{X_0, \ldots, X_t\} = V\}.$$

Show that $\mathbb{E}_a[\tau_{\text{cov}}] \leqslant 2(n-1)|E|$.

(Hint: Consider a spanning tree.)

(c) Show that for all vertices a and b the expected commute time $\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a]$ is at least twice the square of their graph distance, i.e.

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] \geqslant 2d_G(a, b)^2.$$

(Standard results from the lecture course should be stated clearly, but may be used without proof.)

4

Let G be an infinite connected unweighted graph.

(a) Define the terms "wired uniform spanning forest" (WSF) and "free uniform spanning forest" (FSF).

(b) Suppose that G is a recurrent graph. Show that WSF=FSF.

Suppose now that G is an infinite connected unweighted tree.

(c) Show that G is transient if and only if there exists an edge e so that upon removal it creates two transient components.

(d) Suppose that WSF=FSF. Show that G is recurrent.

END OF PAPER