UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 1:30 pm to 3:30 pm

PAPER 213

STOCHASTIC NETWORKS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay on mathematical models of loss networks. Your essay should cover the following topics, but need not be restricted to them.

(i) The stationary distribution for a loss network operating under fixed routing.

(ii) The Erlang fixed point approximation for a loss network, including its existence and uniqueness when routing is fixed.

(iii) An example of a loss network with alternative routing where the Erlang fixed point approximation is not unique.

2

Define a *Wardrop equilibrium* for the flows in a congested network.

Show that if the delay $D_j(y_j)$ at link j is a continuous, strictly increasing function of the throughput, y_j, of link j then a Wardrop equilibrium exists and solves an optimisation problem of the form

$$\begin{aligned} \text{minimise} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)\mathrm{d}u \\ \text{over} \quad & x \geqslant 0,\ y \\ \text{subject to} \quad & Hx = f, \quad Ax = y\ , \end{aligned}$$

where $f = (f_s, s \in S)$ and f_s is the (fixed) aggregate flow between source-sink pair s. What is the interpretation of the matrices A and H? Are the equilibrium throughputs, y_j, unique? Are the equilibrium flows, x_s, unique? Justify your answers.

Suppose now that the aggregate flow between source-sink pair s is not fixed, but is a continuous, strictly decreasing function $B_s(\lambda_s)$, where λ_s is the minimal delay over all routes serving the source-sink pair s, for each $s \in S$. For the extended model, show that an equilibrium exists and solves the optimisation problem

$$\begin{aligned} \text{minimise} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)\mathrm{d}u - G(f) \\ \text{over} \quad & x \geqslant 0,\ y,\ f \\ \text{subject to} \quad & Hx = f, \quad Ax = y\ , \end{aligned}$$

for a suitable choice of the function $G(f)$, to be determined. Are the equilibrium source-sink flows, f_s, unique?

3

Briefly outline a mathematical model of a slotted infinite-population random access scheme, where N_t is the number of stations with a packet to transmit and each such station independently transmits its packets with probability $1/S_t$. Interpret the equation

$$N_{t+1} = N_t + Y_t - I[Z_t = 1]$$

where $Z_t = 0, 1$ or $*$ according as $0, 1$ or more than 1 packets are transmitted in slot $(t, t+1)$ and Y_t is the number of arrivals in slot $(t, t+1)$, assumed to have a Poisson distribution with mean ν.

Suppose that S_t is updated by the recursion

$$S_{t+1} = \max\{1, S_t + aI[Z_t = 0] + bI[Z_t = 1] + cI[Z_t = *]\}$$

for a triplet (a, b, c). Briefly explain why (S_t, N_t) is a Markov chain.

Motivate the differential equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (a-c)e^{-\kappa} + (b-c)\kappa e^{-\kappa} + c, \qquad \frac{\mathrm{d}n}{\mathrm{d}t} = \nu - \kappa e^{-\kappa}$$

where $\kappa = n/s$ in terms of the expected drift of (N_t, S_t).

Show that if $(a, b, c) = (2 - e, 0, 1)$ then, provided $\nu < e^{-1}$, any trajectory solving the differential equations converges to the origin. What happens if $\nu > e^{-1}$?

Show that the Markov chain (S_t, N_t) is transient whatever the choice of (a, b, c) if $\nu > e^{-1}$.

4

Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation and describe its relation to the solution of an optimisation problem.

Consider a network of resources $J = \{1, 2, 3, 4\}$, each of unit capacity, and routes $R = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation. Show, in particular, that if $n_{\{1,2\}} > 0$ then

$$x_{\{1,2\}} n_{\{1,2\}} = \frac{n_{\{1,2\}} + n_{\{3,4\}}}{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}.$$

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rate $\rho_r, r \in R$, and that document sizes are independent and exponentially distributed with unit mean for each route $r \in R$. Determine the transition rates of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B^{-1} \binom{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}{n_{\{1,2\}} + n_{\{3,4\}}} \prod_{r \in R} \rho_r^{n_r},$$

provided it exists.

END OF PAPER