UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2018 9:00 am to 12:00 pm

PAPER 211

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

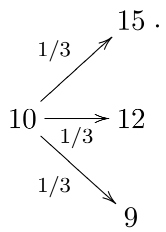
Consider a one-period n-asset market model with the price of asset i at time $t \in \{0, 1\}$ denoted P_t^i.

(a) What is an arbitrage strategy?

(b) What is a pricing kernel?

(c) Show that there is no arbitrage if there exists a pricing kernel.

Now consider the case where $n = 3$. The first asset is cash, so that $P_t^1 = 1$ for all t. The second asset is a stock with price $P_t^2 = S_t$ given by



(The diagram should be read $S_0 = 10$ and $\mathbb{P}(S_1 = 15) = 1/3$, etc. The third asset is a put option written on the stock with maturity $T = 1$, strike $K = 11$ and initial price $P_0^3 = \xi_0$.

(d) What are the possible values of ξ_0 such that the market has no arbitrage?

(e) Suppose that $\xi_0 = 1$. Find the set of all arbitrages, and the subset of pure-investment (i.e. no initial consumption) arbitrages.

2

Suppose that $X = (X_t)_{t \geqslant 0}$ is a discrete-time local martingale.

(a) Show that if X is integrable, i.e. $\mathbb{E}(|X_t|) < \infty$ for all $t \geqslant 0$, then X is a true martingale.

(b) Show that if X is non-negative, then X is a true martingale.

(c) Let $K = (K_t)_{t \geqslant 1}$ be a predictable process. Let $M_0 = 0$ and

$$M_t = \sum_{s=1}^{t} K_s (X_s - X_{s-1})$$

for $t \geqslant 1$. Show that M is a local martingale.

(d) Suppose X is a non-negative super martingale and define a stopping time by $\tau = \inf\{t \geqslant 0 : X_t = 0\}$. Show that $X_t = 0$ on the event $\{t \geqslant \tau\}$.

3

A discrete-time Markov process $(X_t)_{t \geqslant 0}$ is called *affine* iff there exist (finite-valued) functions A and B such that

$$\log \mathbb{E}[e^{\theta X_1} | X_0 = x] = A(\theta)x + B(\theta)$$

for all real θ and x.

(a) Let $(\xi_t)_{t \geqslant 1}$ be an independent and identically distributed sequence, and let

$$\log \mathbb{E}[e^{\theta \xi_1}] = \psi(\theta)$$

for all real θ where ψ is a given function. Let the Markov process X evolve via the equation

$$X_t = aX_{t-1} + b + \xi_t$$

where a and b are given constants. Show that X is affine.

(b) Let X be an affine process. Show that for all $t \geqslant 1$ there are functions A_t and B_t such that

$$\log \mathbb{E}[e^{\theta_1 X_1 + \ldots + \theta_t X_t} | X_0 = x] = A_t(\theta_1, \ldots, \theta_t)x + B_t(\theta_1, \ldots, \theta_t)$$

for all $\theta_1, \ldots, \theta_t$ and x.

(c) Suppose X is the affine process defined in part (a). Find explicit formulae for the functions A_t and B_t defined in part (b), in terms of the parameters a, b and the function ψ.

(d) Consider a discrete-time financial market model where the unique martingale deflator Y evolves via the equation

$$Y_t = e^{X_t} Y_{t-1}$$

where X is an affine process. Show that there exist functions α and β such that, for any $0 \leqslant t \leqslant T$, the time-$t$ price $P_{t,T}$ of a zero-coupon bond of maturity T is given by

$$P_{t,T} = e^{\alpha(T-t)X_t + \beta(T-t)}.$$

4

Let $(S_t)_{t \geqslant 0}$ be a discrete-time martingale such that S_0 is an integer and for all $t \geqslant 1$ the increment $S_t - S_{t-1}$ is valued in the set $\{-1, 0, 1\}$.

(a) Prove the identity

$$(S_T - K - 1)^+ - 2(S_T - K)^+ + (S_T - K + 1)^+ = \mathbf{1}_{\{S_T = K\}}$$

for integers K and $T \geqslant 0$.

(b) Prove the identity

$$(S_T - K)^+ = (S_0 - K)^+ + \sum_{t=1}^{T} f(S_{t-1} - K)(S_t - S_{t-1}) + \frac{1}{2}\sum_{t=1}^{T} \mathbf{1}_{\{S_{t-1} = K\}}(S_t - S_{t-1})^2$$

for integers K and $T \geqslant 1$, where f is defined by

$$f(x) = \mathbf{1}_{\{x > 0\}} + \frac{1}{2}\mathbf{1}_{\{x = 0\}}.$$

Let

$$C(T, K) = \mathbb{E}[(S_T - K)^+]$$

for integers K and $T \geqslant 0$ and

$$\sigma^2(T, K) = \mathrm{Var}(S_{T+1} | S_T = K)$$

for integers K and T such that $|K - S_0| \leqslant T$.

(c) Using parts (a) and (b), or otherwise, prove the identity

$$C(T+1, K) - C(T, K) = \frac{1}{2}\sigma^2(T, K)[C(T, K+1) - 2C(T, K) + C(T, K-1)]$$

for integers K and T such that $|K - S_0| \leqslant T$.

(d) Show that the transition probabilities $\mathbb{P}(S_{T+1} = H | S_T = K)$ for integers K and T such that $|K - S_0| \leqslant T$ can be recovered from the function $C(\cdot, \cdot)$.

5

Let ξ be a random variable with finite exponential moments. Define two functions

$$C(k) = \mathbb{E}[(e^{\xi} - e^{k})^{+}] \text{ for real } k$$

and

$$M(z) = \mathbb{E}[e^{z\xi}] \text{ for complex } z.$$

(a) Show that the identity

$$M(z) = \int_{-\infty}^{\infty} C(k) f(z,k) dk$$

holds for all complex $z = x + \mathrm{i}y$ with $x > 1$, where $f(z,k) = z(z-1)e^{(z-1)k}$.

(b) Show that the identity

$$C(k) = \frac{1}{2\pi \mathrm{i}} \int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} \frac{M(z)}{f(z,k)} dz$$

holds for all real k and $x_0 > 1$.

[You may assume a complex path integral can be computed as a Lebesgue integral by the formula

$$\int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} h(z) dz = \mathrm{i} \int_{-\infty}^{+\infty} h(x_0 + \mathrm{i}y) dy.$$

Also, you may use the following identity without proof:

$$\frac{1}{2\pi \mathrm{i}} \int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} \frac{e^{az}}{z(z-1)} dz = (e^{a} - 1)^{+}$$

for real a and real $x_0 > 1$.]

(c) Consider a continuous-time market model with three assets. The first asset is cash, so the risk-free interest rate is zero.

The second asset is a stock whose time-t price S_t evolves as

$$dS_t = S_t \sigma_t dW_t^S$$

where W^S is a Brownian motion. Finally, the spot volatility process $(\sigma_t)_{t \geqslant 0}$ is bounded and satisfies the stochastic differential equation

$$d\sigma_t = A(\sigma_t)dt + B(\sigma_t)dW_t^{\sigma}$$

where A and B are given functions and W^{σ} is another Brownian motion with $\langle W^S, W^{\sigma} \rangle_t = \rho t$ for a given correlation parameter ρ.

The third asset is a call option with strike K and maturity T. The call's time-t price C_t is computed as follows. Suppose that for all complex z, the bounded function $U(\cdot, \cdot, z)$ satisfies the partial differential equation

$$\frac{1}{2}\sigma^2 z(z-1)U + \frac{\partial U}{\partial t} + (A(\sigma) + z\sigma B(\sigma)\rho)\frac{\partial U}{\partial \sigma} + \frac{1}{2}B(\sigma)^2 \frac{\partial^2 U}{\partial \sigma^2} = 0$$

with terminal condition $U(T,\sigma,z) = 1$ for all σ and z. Set

$$C_t = \frac{1}{2\pi \mathrm{i}} \int_{x_0 - \mathrm{i}\infty}^{x_0 + \mathrm{i}\infty} \frac{S_t U(t,\sigma_t,z)}{f(z, \log(K/S_t))} dz$$

for a fixed $x_0 > 1$. Show that the market has no arbitrage. [You may use standard results of stochastic calculus, such as Itô's formula, without proof. You may also assume that any local martingale appearing in your calculation is a true martingale. Finally, you may use any fundamental theorems of asset pricing without proof.]

6

Consider a two asset model with price dynamics

$$\begin{aligned} dB_t &= B_t r_t dt \\ dS_t &= S_t(\mu_t dt + \sigma_t dW_t) \end{aligned}$$

where r, μ, σ are bounded continuous processes such that $\sigma_t(\omega) > 0$ for all (t, ω) and where W is a Brownian motion. Suppose all processes are adapted to the filtration generated by W.

(a) Show that there is a unique local martingale deflator $Y = (Y_t)_{t \geqslant 0}$ with $Y_0 = 1$, and that its dynamics are of the form

$$dY_t = -Y_t(r_t dt + \lambda_t dW_t)$$

where the process λ is to be expressed in terms of the processes r, μ and σ.

(b) Let $X = (X_t)_{t \geqslant 0}$ be the wealth of a self-financing investor. Assume that the investor does not consume and that the wealth is always non-negative. Show that the process XY is a supermartingale.

(c) Let ξ_T be the payout of a European contingent claim with maturity $T > 0$. Assume ξ_T is non-negative and bounded. Show that the investor can replicate the payout of the claim. Show that the minimal initial cost of the replicating strategy is $\mathbb{E}[Y_T \xi_T]$. [You may use standard results from stochastic calculus.]

(d) Now assume the processes r, σ, μ are constant and that the claim has payout $\xi_T = \sqrt{S_T}$. Find the minimal cost replicating strategy. [You may use without proof standard results from continuous-time financial models.]

END OF PAPER