UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 9:00 am to 11:00 am

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{C}$ denote the complex plane, $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{im}(z) > 0\}$ denote the upper half-plane, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk.

(a) Give the definition of a compact $\mathbb{H}$-hull A and its half-plane capacity $\operatorname{hcap}(A)$.

(b) Let $A = (\mathbb{H} \cap \overline{\mathbb{D}}) \cup ((0, 2i])$. Compute $\operatorname{hcap}(A)$.

(c) State what it means for a family of compact $\mathbb{H}$-hulls to satisfy the conformal Markov property.

(d) Prove that if a family of compact $\mathbb{H}$-hulls satisfies the conformal Markov property, then the Loewner driving function U must be a non-negative multiple of a Brownian motion.

(e) Consider the chordal Loewner equation with driving function $U_t = a\sqrt{t}$ where $a \in \mathbb{R}$ is a constant and let (A_t) be the associated family of compact $\mathbb{H}$-hulls. Prove that $L = \cup_t A_t$ is a straight line from 0 to ∞. *[You do not need to identify the angle of L.]*

2

(a) State (without proof) for which range of κ values SLE_κ is:

(i) Simple,

(ii) Self-intersecting but not space-filling,

(iii) Space filling.

(b) Suppose that X is a Bessel process of dimension δ. Prove that $X^{2-\delta}$ is a continuous local martingale. Prove that if $\delta > 2$ and $X_0 > 0$ then $\inf_{t \geqslant 0} X_t > 0$ almost surely.

(c) Suppose that (g_t) is the Loewner flow with driving function $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion. Show that $(g_t(x) - U_t)/\sqrt{\kappa}$ is a Bessel process and identify its dimension. Prove that SLE_κ is simple for the range of κ values identified in (i) of part (a).

(d) Let γ be an SLE_κ curve in $\mathbb{H}$ from 0 to ∞. For each $r > 0$, let $\sigma_r = \inf\{t \geqslant 0 : \operatorname{Im}(\gamma(t)) = r\}$. Explain why $\mathbb{P}[\sigma_r < \infty]$ does not depend on r. Deduce that $\mathbb{P}[\sigma_r < \infty] = 1$ for every $r > 0$. *[You may not assume the transience of SLE_κ.]*

3 We assume throughout that $D \subseteq \mathbb{C}$ is a simply connected domain distinct from $\mathbb{C}$ and $\emptyset$.

(a) Give the definitions of:

(i) the Dirichlet inner product,

(ii) the space $H_0^1(D)$,

(iii) the Gaussian free field h on D.

(b) Let $G(x, y) = -\log|x - y| - G_x(y)$ where $G_x(y)$ is the function which is harmonic in D with boundary values $y \mapsto -\log|x - y|$. Explain how the L^2 inner product (h, ϕ) is defined for $\phi \in C_0^\infty(D)$ and show that it is a mean-zero normal random variable with variance $\iint \phi(x)G(x, y)\phi(y)dxdy$. *[You may assume without proof that* $-2\pi\Delta^{-1}\phi(x) = \int G(x, y)\phi(y)dy$.*]*

(c) Suppose that ϕ is a conformal transformation $\mathbb{D} \to D$ with $\phi(0) = z$. Explain why the map $\psi \colon \mathbb{D} \to \mathbb{R}$ given by

$$w \mapsto \begin{cases} \log|\frac{\phi(w)-\phi(0)}{w}| & \text{for} \quad w \neq 0 \\ \log|\phi'(0)| & \text{for} \quad w = 0 \end{cases}$$

is harmonic in $\mathbb{D}$. Deduce that

$$\log|\phi'(0)| = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \log|\phi(w) - \phi(0)|dw$$

where dw denotes the Lebesgue measure on $\partial\mathbb{D}$.

(d) Explain why $G_z(y)$ is equal to $-\psi(\phi^{-1}(y))$ and use this to prove that $G_z(z) = -\log|\phi'(0)|$.

4

(a) State what it means for SLE_6 to satisfy the locality property. State what it means for $\mathrm{SLE}_{8/3}$ to satisfy the restriction property.

(b) Fix $T > 0$ and let $D \subseteq \mathbb{H}$ be a simply connected domain. Suppose that $(A_t)_{t\in[0,T]}$ is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing with $A_0 = 0$, $\mathrm{hcap}(A_t) = 2t$ for all $t \in [0,T]$, and $A_T \subseteq D$. Let $\psi\colon D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Show that the family of compact $\mathbb{H}$-hulls $\widetilde{A}_t = \psi(A_t)$ for $t \in [0,T]$ is locally growing with $\widetilde{A}_0 = \emptyset$ and with

$$\mathrm{hcap}(\widetilde{A}_t) = \int_0^t 2(\psi_s'(U_s))^2 ds \quad \text{where} \quad \psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1} \quad \text{for each} \quad t \in [0,T]$$

and $\widetilde{g}_t$ is the unique conformal transformation $\mathbb{H} \setminus \widetilde{A}_t \to \mathbb{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$.

(c) Give the definition of a Brownian excursion $\widehat{B}$ in $\mathbb{H}$ from 0 to ∞. Suppose that A is a compact $\mathbb{H}$-hull with $0 \notin \partial A$. Prove that $\mathbb{P}[\widehat{B}([0,\infty)) \cap A = \emptyset] = \psi_A'(0)$ where $\psi_A\colon \mathbb{H} \setminus A \to \mathbb{H}$ is the unique conformal transformation with $\psi_A(0) = 0$ and $\psi_A(z)/z \to 1$ as $z \to \infty$. *[You may use estimates for the maps g_A provided you state them clearly.]*

END OF PAPER