UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2018 1:30 pm to 3:30 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Stochastic Calculus and Applications

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space.

(i) Define what it means for a process H to be previsible.

Let H be an adapted, left-continuous process. Must H then be previsible?

Let H be an adapted, càdlàg process. Must H then be previsible?

(ii) Let X be an adapted finite variation process, and let H be a previsible process. Define $H \cdot X$ and prove that $H \cdot X$ is a adapted process.

(iii) Let B be a standard one-dimensional Brownian motion. For $f : [0, \infty) \to \mathbb{R}$ a step function of the form $f = \sum_{i=1}^n f_i 1_{(s_{i-1}, s_i]}$ for deterministic $0 \leqslant s_1 < \cdots < s_n < \infty$ and $f_i \in \mathbb{R}$, define the stochastic integral

$$Z_t = \int_0^t f(s)\, dB_s.$$

Prove that (Z_t) is a Gaussian process, i.e., that $(Z_{t_1}, \ldots, Z_{t_n})$ is an n-dimensional Gaussian vector for all $t_1 < \cdots < t_n$, $n \in \mathbb{N}$. Find its covariance function $C(s,t) = \mathrm{cov}(Z_s, Z_t)$. You may not use any properties of the Itô integral unless you prove them.

Explain how this definition can be extended to $f \in L^2(\mathbb{R}_+)$.

2 Stochastic Calculus and Applications

Let B and $\tilde{B}$ be two independent Brownian motions.

(i) Calculate $\langle B, \tilde{B} \rangle_t$ for all $t \geqslant 0$.

(ii) Let $\rho \in [-1, 1]$ and set $W_t = \rho B_t + \sqrt{1-\rho^2}\tilde{B}_t$. Show that W is a Brownian motion and compute $\langle W, B \rangle_t$ for all $t \geqslant 0$.

(iii) Determine which of the following processes are local martingales:

$$X_t = e^{\frac{1}{2}t} \cos B_t, \qquad X_t = B_t - t^2.$$

Are they martingales?

(iv) Let $h : \mathbb{R} \to \mathbb{R}$ be smooth with compact support. Consider the SDE

$$dX_t = h'(X_t)\, dt + dB_t$$

with $X_0 = x \in \mathbb{R}$. Show that for any bounded measurable $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_x(f(X_t)) = \mathbb{E}_x \left(\exp \left(h(B_t) - h(x) - \int_0^t V(B_s)\, ds \right) f(B_t) \right),$$

where

$$V(y) = \frac{1}{2} h'(y)^2 + \frac{1}{2} h''(y).$$

[*You may use any results proved in the lectures*]

3 Stochastic Calculus and Applications

(i) Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions. Prove that there is pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

(ii) Let $b \in \mathbb{R}$, $\sigma > 0$, $x \in \mathbb{R}$ be constants, and let X be the solution of

$$X_t = x + b\int_0^t X_s ds + \sigma \int_0^t X_s dB_s.$$

Find $\mathbb{E}(X_t^k)$ for all natural numbers k.

(iii) Let X_0 be a standard normal random variable and suppose that

$$dX_t = -\frac{1}{2}X_t\, dt + dB_t.$$

X_0 is independent of the Brownian motion. Find the distribution of X_t for $t \geqslant 0$ and find $\mathrm{cov}(X_t, X_s)$ for all t, s.

4 Stochastic Calculus and Applications

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space.

(i) Let X be a local martingale.

Show that if X is nonnegative then X is a supermartingale.

Suppose that X is bounded. Is X then a martingale? If you use any result from the Stochastic Calculus lectures, you must prove it.

(ii) Give an example of a local martingale that is not a martingale. You do not need to show that the sequence of stopping times you use tends to infinity.

(iii) Let B be a standard one-dimensional Brownian motion, and let H be a continuous, adapted, bounded process. Prove that

$$\frac{\int_t^{t+h} H_s\, dB_s}{B_{t+h} - B_t} \to H_t \quad \text{in probability as } h \downarrow 0.$$

Hint: Estimate $\mathbb{E}(|B_{t+h} - B_t|^{-1/2})$ and show that

$$\mathbb{E}\left(\left|\int_t^{t+h} (H_s - H_t)\, dB_s\right|^{1/2}\right) \leqslant \mathbb{E}\left(\int_t^{t+h} (H_s - H_t)^2\, ds\right)^{1/4}.$$

5 Stochastic Calculus and Applications

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space. Let M be a continuous local martingale with $M_0 = 0$.

(i) Let $a, b > 0$. Prove that

$$\mathbb{P}(\sup_{s \leqslant t} M_s > a) \leqslant \frac{4b}{a^2} + \mathbb{P}(\langle M \rangle_t > b).$$

(ii) Show that there is a sequence of stopping times (S_n) such that for each n, the stopped process M^{S_n} is a bounded continuous martingale.

(iii) For $p > 0$, define

$$Z_n = \sum_{i=1}^{2^n} |M_{2^{-n}i} - M_{2^{-n}(i-1)}|^p.$$

Let $p > 2$. Show that $Z_n \to 0$ in probability as $n \to \infty$.

Let $1 < p < 2$ and assume that $\limsup_{n \to \infty} Z_n < \infty$ almost surely. Show that M is indistinguishable from 0 on $[0, 1]$.

(iv) Show that there can be at most one finite variation process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a continuous local martingale.

Is it also always true that such a $\langle M \rangle$ exists? Explain your answer in two sentences.

6 Stochastic Calculus and Applications

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space, and let X, Y be continuous local martingales. The Stratonovich integral of X with respect to Y is defined by

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s \, dX_s + \frac{1}{2}\langle X, Y\rangle_t,$$

where the first term on the right-hand side is a usual Itô integral.

(i) For every $f \in C^3$ show that

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) \circ dX_s.$$

(ii) Show that $(\int_0^t X_s \circ dY_s)_t$ is in general not a local martingale.

(iii) Show that

$$\int_0^t Y_s \circ dX_s = \lim_{n\to\infty} \sum_{i=1}^{\lceil t2^n \rceil} \frac{1}{2}(Y_{2^{-n}i} + Y_{2^{-n}(i-1)})(X_{2^{-n}i} - X_{2^{-n}(i-1)}).$$

[*You may use results proved in the lectures provided they are clearly stated.*]

(iv) The Hermite polynomials h_n are defined by

$$h_n(x) = e^{x^2/2}(-1)^n \frac{d^n}{dx^n} e^{-x^2/2}.$$

Let $H_n(x,t) = t^{n/2} h_n(x/\sqrt{t})$ for $t > 0$ and $H_n(x,0) = x^n$. You may use that

$$\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial H_n}{\partial t} = 0, \quad \frac{\partial H_n}{\partial x} = nH_{n-1} \ \ (n \geqslant 1).$$

For all n, show that $H_n(B_t, t)$ is a local martingale, where B is a standard Brownian motion. Compute $H_n(B_t, t)$ for $n = 1, 2, 3$.

END OF PAPER