UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2018 9:00 am to 12:00 pm

PAPER 201

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State Doob's upcrossing lemma for discrete time martingales.

(b) State and prove the almost sure martingale convergence theorem for discrete time martingales.

(c) Let $(X_n)_{n\geqslant 0}$ be a sequence of independent random variables with $\mathbf{E}X_n = 0$ for all n, and $\sum_{n=0}^{\infty} \mathbf{E}X_n^2 < \infty$. Show that $\sum_{n=0}^{\infty} X_n$ converges almost surely.

(d) Give an example of a martingale that converges almost surely but not in L^1.

2

(a) Define *uniform integrability* of a sequence of random variables $(X_n)_{n\geqslant 0}$, and state the convergence theorem for uniformly integrable discrete time martingales.

(b) Let $(X_n)_{n\geqslant 0}$ be a discrete time martingale, and let T be an almost surely finite stopping time such that,

$$\mathbf{E}|X_T| < \infty \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\big(|X_n|\mathbf{1}_{\{T>n\}}\big) = 0.$$

Prove that the stopped process $(X_{n\wedge T})_{n\geqslant 0}$ is uniformly integrable.

(c) Let T be a random variable taking values in the natural numbers $\{0,1,2,\dots\}$. Prove that

$$\mathbf{E}T = \sum_{n=0}^{\infty} \mathbf{P}(T > n).$$

(d) Let $(X_n)_{n\geqslant 0}$ be a discrete time martingale with respect to a filtration $(\mathcal{F}_n)_{n\geqslant 0}$ such that $X_0 = 0$ and for all $n \geqslant 0$,

$$\mathbf{E}(|X_{n+1} - X_n| \mid \mathcal{F}_n) \leqslant C \quad \text{a.s.}$$

for some finite constant C. Let T be a stopping time with $\mathbf{E}T < \infty$. Prove that $\mathbf{E}X_0 = \mathbf{E}X_T$.

[Hint: Show that $Y = \sum_{n=0}^{\infty} |X_{n+1} - X_n|\mathbf{1}_{\{T>n\}}$ is integrable and use it to bound the quantities appearing in (b).]

3 Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed and integrable random variables with $\mathbf{E}X_1 = m$. For $n \geqslant 1$, let $S_n = X_1 + \cdots + X_n$.

(a) Define the *tail σ-algebra* of $(X_n)_{n\geqslant 1}$ and state Kolmogorov's 0-1 law.

(b) Show that

$$\mathbf{E}(X_1 \mid S_n, S_{n+1}, \ldots) = \frac{S_n}{n} \quad \text{a.s.}$$

(c) Prove the strong law of large numbers, i.e.,

$$\lim_{n\to\infty} \frac{S_n}{n} = m \quad \text{a.s.}$$

[You need not prove the backwards martingale convergence theorem.]

(d) Let

$$\tilde{S}_n = X_1X_2 + X_2X_3 + \cdots + X_nX_{n+1}.$$

Show that

$$\lim_{n\to\infty} \frac{\tilde{S}_n}{n} = m^2 \quad \text{a.s.}$$

4

(a) Define a *standard Brownian motion* $(B_t)_{t\geqslant 0}$ in one dimension.

(b) Consider the process $U_t = B_t - tB_1$ for $t \in [0,1]$ and show that $(U_t)_{t\in[0,1]}$ is independent of B_1.

(c) Consider the process $W_t = (B_t, B'_t)$ for $t \geqslant 0$, where $(B'_t)_{t\geqslant 0}$ is an independent version of $(B_t)_{t\geqslant 0}$. Let $A \subset \mathbb{R}^2$ be given by

$$A = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x - y \leqslant 3\}$$

and let $\tau = \inf\{t \geqslant 0 : W_t \in \partial A\}$. Compute $\mathbf{P}(B_\tau = B'_\tau - 1)$.

(d) Let $(\Omega, \mathbf{P})$ be the probability space on which $(B_t)_{t\geqslant 0}$ is defined. Prove that

$$\mathbf{P}(\{\omega \in \Omega : t \mapsto B_t(\omega) \text{ is uniformly continuous on } [0,\infty)\}) = 0.$$

5 Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion in one dimension and let $S_t = \sup_{0\leqslant s\leqslant t} B_s$.

(a) State the reflection principle for $(B_t)_{t\geqslant 0}$ and a stopping time T.

(b) Show that $\mathbb{P}(S_\infty = \infty) = 1$.

(c) Prove that for any $x \leqslant y$, $y > 0$,

$$\mathbf{P}(S_t \geqslant y, B_t \leqslant x) = \mathbf{P}(B_t \geqslant 2y - x).$$

(d) A random variable T is said to have an α-*stable distribution* if for all $n \in \mathbb{N}$,

$$\frac{T^{(1)} + T^{(2)} + \ldots + T^{(n)}}{n^{1/\alpha}} \sim T,$$

where $T^{(i)}$ are independent copies of T and $\sim$ denotes equality in distribution. Show that $T_a = \inf\{s \geqslant 0 : B_s \geqslant a\}$, $a > 0$, has a $\frac{1}{2}$-stable distribution.
[Hint: Take $x = y = a$.]

6

(a) State Lévy's continuity theorem.

(b) We say that $(X_n)_{n\geqslant 1}$ converges in distribution if the laws of X_n converge weakly.

Let $(X_n)_{n\geqslant 1}$ be independent and identically distributed. Assume that for all $t \in \mathbb{R}$,

$$\mathbf{E}e^{itX_1} = 1 + iat + f(t)$$

where a is a constant, and

$$\frac{|f(t)|}{t} \to 0 \qquad \text{as } t \to 0, t \neq 0.$$

Show that

$$\frac{X_1 + \ldots + X_n}{n} \to a \quad \text{in distribution as } n \to \infty.$$

(c) We say that a sequence of real random variables $(X_n)_{n\geqslant 0}$ defined on the same probability space $(\Omega, \mathbf{P})$ converges in probability to a random variable X if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

Prove that if $(X_n)_{n\geqslant 0}$ converges in probability, then it converges in distribution, and give a counterexample for the converse statement.

(d) Prove that if $(X_n)_{n\geqslant 0}$ converges in distribution to 0, then it converges in probability to 0.

END OF PAPER