UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2018 9:00 am to 11:00 pm

PAPER 142

CHARACTERISTIC CLASSES AND K-THEORY

You must attempt Question 1, and you may attempt at most ***ONE*** *further question,*

There are ***FOUR*** *questions in total.*

Question 1 is worth 40 marks; the remaining questions are each worth 30 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a compact Hausdorff space. If $L_1 \to X$ and $L_2 \to X$ are complex line bundles, prove that the first Chern class satisfies

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(X;\mathbb{Z}).$$

State the splitting principle in cohomology for complex vector bundles. Define the Chern character $\mathrm{ch} : K^0(X) \to H^{2*}(X;\mathbb{Q})$, and prove that it is a ring homomorphism. *[You may use any results from the theory of symmetric polynomials.]*

Hence compute the rings $K^0(\mathbb{CP}^n)$. *[You may use any properties of the K-theory of spheres.]*

2 Let X be a compact Hausdorff space and $\pi : E \to X$ be a real vector bundle. State the projective bundle formula in cohomology for real vector bundles, and using this define the Stiefel–Whitney classes $w_i(E) \in H^i(X;\mathbb{F}_2)$. If $\pi' : E' \to X$ is another real vector bundle, prove that

$$w_k(E \oplus E') = \sum_{i+j=k} w_i(E) \smile w_j(E') \in H^k(X;\mathbb{F}_2).$$

Let $f : \mathbb{RP}^{2k+1} \to \mathbb{CP}^k$ be a continuous map which is non-trivial on second $\mathbb{F}_2$-cohomology. If the map

$$f \times f : \mathbb{RP}^{2k+1} \longrightarrow \mathbb{CP}^k \times \mathbb{CP}^k$$

is homotopic to an immersion, show that k must be odd. *[You may use without proof any description of the cohomology and tangent bundles of real and complex projective spaces, providing they are clearly stated.]*

3 Describe the abelian groups $K^*(S^n)$, and the ring structure on $K^0(S^n)$. If X is a finite CW-complex, prove by induction over the number of cells of X that

1. the Euler characteristic $\chi(X)$ of X is equal to $\mathrm{rank}\, K^0(X) - \mathrm{rank}\, K^{-1}(X)$,
2. every element of $\widetilde{K}^0(X)$ is nilpotent.

If $p : Y \to X$ is a n-fold covering space of a finite CW-complex, construct a homomorphism

$$p_! : K^0(Y) \longrightarrow K^0(X)$$

and use it to show that the map $p^* : K^0(X) \otimes \mathbb{Z}[\frac{1}{n}] \to K^0(Y) \otimes \mathbb{Z}[\frac{1}{n}]$ is injective.

4 What is the K-theory *Euler class* of a complex vector bundle $\pi : E \to X$? What is the *Gysin sequence* in K-theory associated to this vector bundle?

Use the Gysin sequence to calculate the ring $K^0(\mathbb{RP}^{2n+1})$ and the abelian group $K^{-1}(\mathbb{RP}^{2n+1})$. *[You may use any description of the K-theory of $\mathbb{CP}^n$ without proof.]*

If $L = \gamma_{\mathbb{R}}^{1,2n+2} \otimes \mathbb{C}$ denotes the complexification of the tautological real line bundle over $\mathbb{RP}^{2n+1}$ and $\pi : E \to \mathbb{RP}^{2n+1}$ denotes the direct sum of k copies of L, compute the abelian group $K^0(\mathbb{S}(E))$.

END OF PAPER