UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 1:30 pm to 4:30 pm

PAPER 141

3-MANIFOLDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $L(5,3)$ denote the $(5,3)$ lens space, with the orientation convention that $L(5,3)$ is the Dehn surgery of slope $5/3$ on the unknot in S^3. Only brief explanations are expected for the following computations. Any results from from lecture or example sheets may be used as long as you state them clearly.

(a) Draw and label a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for the lens space $L(5,3)$, and briefly describe, as a series of handle-attachments to $\Sigma \times I$, the construction of $L(5,3)$ specified by this Heegaard diagram, for I the closed interval $I = [0,1]$.

(b) Write down a presentation for the fundamental group $\pi_1(L(5,3))$ of $L(5,3)$, as determined by the Heegaard diagram from part (a).

(c) Classify, up to oriented homeomorphism, the connected closed oriented 3-manifolds M with $\pi_1(M)$ isomorphic to $\pi_1(L(5,3))$.

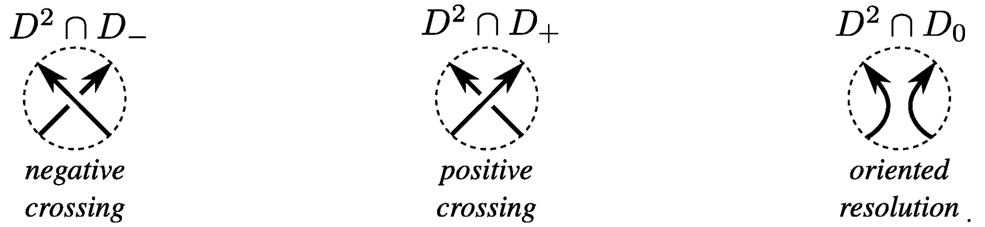
(d) Draw the diagram of a k-component link $L \subset S^3$, for some k, with components labeled by *integer* Dehn surgery coefficients $(n_1, \ldots, n_k) \in \mathbb{Z}^k$, such that Dehn surgery of slope $(n_1, \ldots, n_k)$ along $L \subset S^3$ yields $L(5,3)$.

2 Similarly to the Alexander polynomial, the Jones polynomial $V(D)(q) \in \mathbb{Z}[q^{-1/2}, q^{1/2}]$ is an invariant of oriented link diagrams D. It is known that $V(D)(q) = V(D')(q)$ whenever the diagrams D and D' represent isotopic oriented links in $L, L' \subset S^3$. Thus, V descends to a well-defined invariant of oriented links, defined by $V(L)(q) := V(D)(q)$ for any oriented link diagram D representing an oriented link $L \subset S^3$.

It is known that the Jones polynomial is uniquely determined by the normalisation $V(U_1)(q) = 1$ for the unknot $U_1 \subset S^3$, together with the skein relation that

$$q^1 V(D_-) - q^{-1} V(D_+) = (q^{1/2} - q^{-1/2}) V(D_0)$$

for any diagrams D_-, D_+, D_0 which form a *skein-related triple.* That is, they coincide in the complement of some disk $D^2 \subset \mathbb{R}^2$, but they intersect D^2 in the neighborhood of a negative crossing, positive crossing, and oriented resolution, respectively, as shown:



[In the following, you may take all of the above statements as given. In addition, you need not specify Reidemeister moves relating diagrams of isotopic links.]

(a) For any two oriented links $L_1, L_2 \subset S^3$, the *disjoint union* $L_1 \amalg L_2 \subset S^3$ embeds L_1 and L_2 into disjoint balls in S^3. Compute the Jones polynomial $V(U_2)$ of the 2-component unlink $U_2 := U_1 \amalg U_1 \subset S^3$.

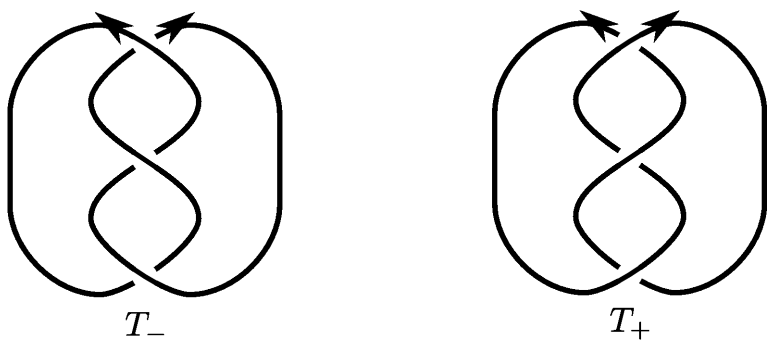
[*Hint: the one-crossing diagrams for the unknot might be useful for part (a).*]

(b) Prove that for any oriented link $L \subset S^3$, the Jones polynomial satisfies

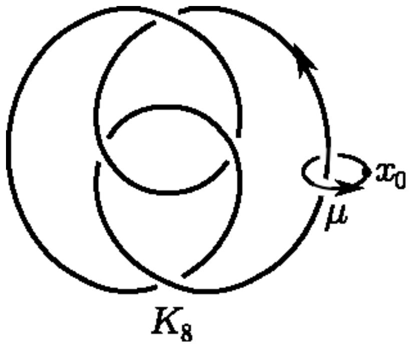
$$V(L \amalg U_1) = V(L)V(U_2).$$

(c) The *mirror* $\overline{L} \subset S^3$ of an oriented link $L \subset S^3$ is given by composing the embedding of L in S^3 with an orientation-reversing homeomorphism of S^3. For any oriented link $L \subset S^3$, briefly argue (i) that reversing all the crossings of a diagram for L (turning each positive crossing into a negative crossing, and vice versa) produces a diagram for $\overline{L}$, and (ii) that $V(\overline{L})(q) = V(L)(q^{-1})$.

(d) Prove that the right-handed trefoil $T_+ \subset S^3$ (shown at right below) is not isotopic to the left-handed trefoil $T_- \subset S^3$ (shown at left below).



3 The questions below refer to the following diagram of the figure-8 knot $K_8 \subset S^3$ with exterior $X_8 := S^3 \setminus \mathring{\nu}(K_8)$, together with the shown choice of meridian $\mu \in \pi_1(X_8, x_0)$.



(a) Draw and label a Dehn presentation Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for X_8, as specified by the knot diagram above. State the corresponding series of handles attached to $\Sigma \times I$ to obtain X_8.

(b) Please (i) write down the (Dehn) presentation of $\pi_1(X_8, x_0)$ associated to this Heegaard diagram, briefly explaining the interpretation of the presentation's generators and relators with respect to the Heegaard diagram, and (ii) compute the images of the generators of this presentation under the abelianization map $\varphi : \pi_1(X_8, x_0) \to H_1(X_8; \mathbb{Z})$, expressing your answers in terms of $\varphi(\mu)$.

(c) For the presentation computed in part (b), compute the Alexander matrix A of Fox derivatives of relators with respect to generators (before taking minors). Express your answer in terms of $t \in \mathbb{Z}[t^{-1}, t] \simeq \mathbb{Z}[H_1(X_8; \mathbb{Z})]$, for $t := [\varphi(\mu)]$ the expression of $\varphi(\mu)$ in multiplicative notation.

(d) Drawing a copy of the above knot diagram for K_8, label *one* Kauffman state of your choice, and circle the entries of the Alexander matrix A whose signed product gives the summand of the Alexander polynomial corresponding to your chosen Kauffman state.

4 We fix the following conventions and terminology for Seifert fibered spaces over S^2.

Regarding $Y :\cong S^1 \times (S^2 \setminus \coprod_{i=1}^k \overset{\circ}{D}{}_i^2)$, with boundary components $\partial_1 Y, \ldots, \partial_k Y$, as the exterior in $S^1 \times S^2$ of k regular fibers, each of class $f = [S^1 \times \{pt\}] \in H_1(Y;\mathbb{Z})$, define $-\tilde{h}_i := [\{pt\} \times \partial D_i^2] \in H_1(\partial_i Y;\mathbb{Z})$ to be the meridian of the i^{th} regular fiber, noting that $\sum_{i=1}^k \tilde{h}_i = 0$, and let $\tilde{f}_i \in H_1(\partial_i Y;\mathbb{Z})$ denote the lift of f (i.e. $\tilde{f}_i \mapsto f$ under the homomorphism $H_1(\partial_i Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ induced by the inclusion $\partial_i Y \hookrightarrow Y$) satisfying $-\tilde{h}_i \cdot \tilde{f}_i = 1$. We say that a homology class $b_i\tilde{f}_i - a_i\tilde{h}_i \in H_1(\partial_i Y;\mathbb{Z})$ has *Seifert slope* $\frac{b_i}{a_i} \in \mathbb{Q} \cup \{\infty\}$. The Seifert fibered space $M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k})$ is then given by the Dehn filling of Y along the Seifert slopes $\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k}$. We lastly fix the notation $\iota_{i*} : H_1(\partial_i Y;\mathbb{Z}) \to H_1(M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k});\mathbb{Z})$ for the homomorphisms induced by inclusion, with images $h_i := \iota_{i*}(\tilde{h}_i)$ and $f := \iota_{i*}(\tilde{f}_i)$, and the notation $M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k}, *) := M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k}) \setminus \overset{\circ}{\nu}(f)$ for the exterior of a regular fiber in $M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k})$.

[In the following, you may state and use any results from lecture or example sheets, as long as they do not specifically concern torus knots.]

(a) Let $p, q, p^*, q^* \in \mathbb{Z}$ be integers satisfying $p, q > 1$ and $pp^* - qq^* = 1$. (i) For $\mu_2 := -q^*\tilde{f}_2 - p\tilde{h}_2$, $\lambda_2 := p^*\tilde{f}_2 + q\tilde{h}_2 \in H_1(\partial_2 M_{S^2}(\frac{p^*}{q}, *);\mathbb{Z})$, show that $\mu_2 \cdot \lambda_2 = 1$ and $\iota_{2*}(\lambda_2) = 0$. As corollaries, show (ii) that $M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p}) \cong S^3$, and (iii) that the regular fiber $f \subset M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p})$ is the (p,q) torus knot $T(p,q) \subset S^3$.

(b) Let $m := -\tilde{h}_3 \in H_1(\partial_3 M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p}, *);\mathbb{Z})$ denote the meridian of the (p,q) torus knot $f \subset M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p}) \cong S^3$, and let λ_1, λ_2 be any longitudes for the exceptional fibers in $M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p})$ with respective meridians $\mu_1 = p^*\tilde{f}_1 - q\tilde{h}_1$ and $\mu_2 = -q^*\tilde{f}_2 - p\tilde{h}_2$. (i) Show that $f = pq\,\iota_{3*}(m) = q\iota_{1*}(\lambda_1) = p\iota_{2*}(\lambda_2)$, and (ii) compute the Turaev torsion $\tau(M_{S^2}(\frac{p^*}{q}, -\frac{q^*}{p}, *))$ of the (p,q) torus-knot exterior.

(c) You make take as given that the torus knot $T(p,q) \subset S^3$ is fibered. Thus its exterior $X_{(p,q)} :\cong S^3 \setminus \overset{\circ}{\nu}(T(p,q))$ can be realised as the mapping torus $X_{(p,q)} \cong M_\phi := (\Sigma \times I)/(x,1) \sim (\phi(x),0)$ of some monodromy homeomorphism $\phi : \Sigma \to \Sigma$, for Σ an appropriate compact surface with boundary. (i) State and justify whether ϕ is isotopic to a reducible, periodic, Anosov, or pseudo-Anosov homeomorphism. (ii) Compute $\det(\phi_* - t(\mathrm{id}))$, for $\phi_* : H_1(\Sigma;\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z})$ the induced homomorphism on first homology. (iii) Compute the genus $g := g(\Sigma)$ of Σ, and justify your answer.

(d) Let $M :\cong X_{(p,q)}(0)$ be the manifold resulting from Dehn surgery of slope 0 on $T(p,q) \subset S^3$, with respect to a conventional surgery basis. For g as determined in part (c.iii), construct a taut foliation $\mathcal{F}$ on M with at least one compact leaf of genus g, and prove that $\mathcal{F}$ is taut.

END OF PAPER