UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 9:00 am to 12:00 pm

PAPER 140

SYMPLECTIC GEOMETRY

*Attempt no more than **FIVE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

All manifolds here are assumed to be closed (no boundary) and connected.

1

(a) State *Moser's theorem.* [You can use it from now on without proving it.]

(b) Prove the following classification result: two compact symplectic surfaces are symplectomorphic if and only if they have the same genus and the same symplectic area. [You may use without proof the fact that two orientable smooth compact surfaces are diffeomorphic if and only if they have the same genus.]

(c) Prove the following theorem, also due to Moser, which is in a sense the higher dimensional analogue of the classification result in part (b):

Theorem. *Let M be a compact n-dimensional manifold and β_0, β_1 be two volume forms on M. Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^*\beta_1 = \beta_0$ if and only if $\int_M \beta_0 = \int_M \beta_1$ (or equivalently $[\beta_0] = [\beta_1] \in H^n_{dR}(M)$).*

2

(a) Let (V, Ω) be a symplectic vector space and let the linear map $S : V \to V$ be an anti-symplectic involution, that is, $S^2 = \mathrm{Id}_V$ and $S^*\Omega = -\Omega$. Show that $\mathrm{Fix}(S)$, the set of points fixed by S, is a Lagrangian subspace of (V, Ω).

(b) In the same setting as above, show that there exists a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ of (V, Ω) such that $Se_i = e_i$ and $Sf_i = -f_i$.

(c) Let (M, ω) be a symplectic manifold and let the diffeomorphism $\sigma : M \to M$ be an anti-symplectic involution, that is, $\sigma^2 = \mathrm{Id}_M$ and $\sigma^*\omega = -\omega$. Show that if it is not empty, then $\mathrm{Fix}(\sigma)$ is a Lagrangian submanifold of (M, ω). [You may use without proof that for each $p \in \mathrm{Fix}(\sigma)$ there are coordinates centered around p in which the map σ is linear.]

(d) Show that $\{(z_1, \dots, z_n) \in \mathbb{C}^n | z_j \in \mathbb{R})\} \simeq \mathbb{R}^n$ is a Lagrangian submanifold of $(\mathbb{C}^n, \widetilde{\omega}_{FS} = \frac{i}{2}\partial\overline{\partial}\ln(|z^2| + 1))$.

(e) Recall that the Fubini–Study form ω_{FS} on $\mathbb{C}P^n = \mathbb{C}^{n+1}/z \sim \lambda z, \lambda \in \mathbb{C} \setminus \{0\}$ is obtained by gluing together $\varphi_i^*\widetilde{\omega}_{FS}$, where $\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i) | \, i = 0, \dots, n\}$ is an atlas for $\mathbb{C}P^n$, with $\mathcal{U}_i = \{[z_0, \dots, z_n] \in \mathbb{C}P^n | z_i \neq 0\}$ and $\varphi_i : \mathcal{U}_i \to \mathbb{C}^n$ is given by $\varphi_i([z_0, \dots, z_n]) = \frac{1}{z_i}(z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$.

Show that $\mathbb{R}P^n = \mathbb{R}^{n+1}/x \sim \lambda x, \lambda \in \mathbb{R} \setminus \{0\}$ is a Lagrangian submanifold of $(\mathbb{C}P^n, \omega_{FS})$.

3 Let (M, ω) be a symplectic manifold. Given an isotopy $\rho_t : M \to M$, let X_t denote the time-dependent vector field that generates it. We say that ρ_t is a symplectic isotopy if each ρ_t is a symplectomorphism and that ρ_t is a Hamiltonian isotopy if X_t is a Hamiltonian vector field for each t. In this problem, (M, ω) is an exact symplectic manifold, i.e., $\omega = -d\lambda$ for some 1-form λ.

(a) Show that ρ_t is a symplectic isotopy if and only if the 1-form $\rho_t^*\lambda - \lambda$ is closed for every t.

(b) In this and the next part, you are asked to show that ρ_t is a Hamiltonian isotopy if and only if the 1-form $\rho_t^*\lambda - \lambda$ is exact for every t.

First, show that if H_t is a family of Hamiltonian functions for the family of vector fields X_t, then $\rho_t^*\lambda - \lambda = dF_t$ where $F_t = \int_0^t (\iota_{X_s}\lambda - H_s) \circ \rho_s \, ds$.

(c) Next, show that if there exists a smooth family of functions $F_t : M \to R$ such that $\rho_t^*\lambda - \lambda = dF_t$ for every t, then X_t is a Hamiltonian vector field for each t.

4 (a) Show that the graph of a 1-form β on X is a Lagrangian submanifold of T^*X if and only if β is closed.

(b) Consider the map $f : T^*X \to T^*X$ given by $f(x, \xi) = (x, \xi + \beta_x)$. Show that if β is closed then f is a symplectomorphism.

(c) Show that if β is exact then f is a Hamiltonian diffeomorphism (i.e. $f = \rho_1$ for some Hamiltonian isotopy ρ_t, as defined in Question 3 above).

5 A submanifold L of an almost complex manifold (M, J) is totally real if $\dim(L) = \frac{1}{2}\dim(M)$ and $T_pL \cap J_p(T_pL) = \{0\}$ for all $p \in L$.

(a) Give the definition of a *compatible almost complex structure* J on (M, ω) and explain what a *compatible triple* (ω, J, g) is.

(b) Let (ω, J, g) be a compatible triple on M. Show that L is a Lagrangian submanifold of (M, ω) if and only if $J_p(T_pL) = (T_pL)^\perp$, where $\perp$ is with respect to g. Conclude that a Lagrangian submanifold L of (M, ω) is a totally real submanifold of (M, J).

(c) Give an example of a totally real submanifold that is not a Lagrangian submanifold (and show that it is indeed an example). Your example need not be compact.

6 Except where explicitly told otherwise, justify all your answers.

(a) Let (M, ω) be a symplectic manifold endowed with a Hamiltonian action of a Lie group G with moment map $\phi : M \to \mathfrak{g}^*$. Let H be a Lie subgroup of G and $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be the inclusion of the Lie algebra. Show that the restriction action of H on (M, ω) is Hamiltonian with moment map given by $\pi \circ \phi : M \to \mathfrak{h}^*$, where $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$ is the projection dual to the above inclusion, i.e., $\langle \pi(\eta), X \rangle_H = \langle \eta, i(X) \rangle_G$.

(b) Consider the diagonal inclusion of the circle S^1 into the torus $\mathbb{T}^n$ given by $t \mapsto (t, \ldots, t)$. In this case, what is the map $\pi : \mathbb{R}^n \to \mathbb{R}$ dual to the diagonal inclusion? (As usual we identify the Lie algebras of S^1 and $\mathbb{T}^n$ and also their duals with $\mathbb{R}$ and $\mathbb{R}^n$ respectively.)

(c) Consider the standard Hamiltonian T^2-action on $(\mathbb{C}P^2, \omega_{FS})$ given by

$$(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0, t_1 z_1, t_2 z_2].$$

What is its moment map $\phi : \mathbb{C}P^2 \to \mathbb{R}^2$? [You can just write the moment map, no justification needed. For the sake of the rest of the question, pick ϕ such that $\phi([1, 0, 0]) = (0, 0)$.] What is the corresponding image $\phi(\mathbb{C}P^2)$?

Now consider the diagonal action of S^1 on $\mathbb{C}P^2$. What points are fixed by it? What is the moment map $\mu : \mathbb{C}P^2 \to \mathbb{R}$? [For the sake of the rest of the question, make sure μ is such that $\mu([1, 0, 0]) = 0$.] What is the corresponding image $\mu(\mathbb{C}P^2)$?

(d) Is the level set $\mu^{-1}(-\frac{1}{4})$ a Lagrangian, or an isotropic, or a coisotropic submanifold of $\mathbb{C}P^2$, or none of these?

(e) What is the quotient space $M_{\text{red}} = \mu^{-1}(-\frac{1}{4})/S^1$ as a manifold? (You are not asked to address its reduced symplectic form ω_{red}.)

END OF PAPER