UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 1:30 pm to 3:30 pm

PAPER 139

POSITIVITY IN ALGEBRAIC GEOMETRY

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) Let X be a projective scheme of dimension n.
State the *Asymptotic Riemann Roch Theorem* for X.

(ii) Let X be a projective scheme of dimension n.
State Nakai's criterion and Kleiman's Theorem for X.

(iii) Let X be a projective scheme of dimension n. Let D be a nef Cartier divisor on X.
Assume that for any projective scheme Y of dimension $k < n$, for any nef Cartier divisor N on Y and for any $0 < i < k$ there exist a positive real number C_i such that

$$h^i(Y, \mathcal{O}_Y(mN)) \leqslant C_i m^{k-1}, \text{ for all } m \gg 0.$$

Show that for any $j > 1$ there exists there exists a positive real number C'_j such that

$$|h^j(X, \mathcal{O}_X(mD))| \leqslant C'_j m^{n-1},$$

for all $m \gg 0$.

[*Hint: as in the proof of Nakai's criterion, you may want to consider the two short exact sequences*

$$0 \to \mathcal{O}_X(mD - H_1) \to \mathcal{O}_X(mD) \to \mathcal{O}_{H_1}(mD) \to 0$$
$$0 \to \mathcal{O}_X(mD - H_1) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{H_2}((m-1)D) \to 0.$$

for a suitable choice of divisors H_1, H_2.]

(iv) Under the same assumptions as in (iii), show that

$$h^0(X, \mathcal{O}_X(mD)) - h^1(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + \text{lower order terms.}$$

Moreover, show that if for any sufficiently positive integer m, $h^0(X, \mathcal{O}_X(mD)) = 0$, then $D^n = 0$ and there exists a positive real number T'_1 such that

$$h^1(X, \mathcal{O}_X(mD)) \leqslant T'_1 m^{n-1}$$

(v) Let X be a projective scheme of dimension n. Show that for any $j \geqslant 1$ there exists there exists a positive real number T_j such that

$$h^j(X, \mathcal{O}_X(mD)) \leqslant T_j m^{n-1}, \text{ for all } m \gg 0.$$

[*Hint: proceed by induction on the dimension of* X *and then use (iii) and (iv) where appropriate. You will need to discuss separately the case where* $h^0(X, \mathcal{O}_X(mD)) \neq 0$ *for infinitely positive values of* m.]

(vi) Let X be a projective irreducible variety of dimension n. Let D be a nef Cartier divisor on D.
Show that

$$\lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n} > 0$$

if and only if $D^n > 0$.

2 In this question X will denote a smooth projective surface defined over an algebraically closed field k.

(i) Let H be a Cartier divisor on X. Assume that $|H| \neq \emptyset$.
Define the *fixed* and the *movable* part of the linear system $|F|$.
Show that:

(a) the movable part of $|H|$ is always nef on X;

(b) if the Kodaira dimension of H is zero, then for any positive integer n the movable part of $|nH|$ is trivial.

(ii) Let D be a nef divisor on X. Assume that $\dim |D| \geqslant 1$ and $D^2 = 0$.
Show that if $|D|$ is movable then D is base point free. Show also that the Kodaira dimension of D is one.

(iii) Let H be a Cartier divisor on X. Assume that $|H| \neq \emptyset$. For any positive integer n, let $|nH| = F_n + |M_n|$ be the decomposition into fixed and movable part.
Show that:

(1) the Kodaira dimension of H is two, if and only if for some n $M_n^2 > 0$.

(2) the Kodaira dimension of H is one if and only if for all n $M_n^2 = 0$ and there exists a positive integer n' such that $\dim |M_{n'}| \geqslant 1$.

For the remainder of the exercise we will assume that on the smooth projective surface X, $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$.

(iv) Show that $\chi(X, \mathcal{O}_X) = 2$.

(v) Let D be a nef divisor on X which is not numerically trivial. Assume that $D^2 = 0$. Then $h^0(X, \mathcal{O}_X(D)) \geqslant 2$.

(vi) Let $|D| = F + |M|$ be the decomposition into fixed and movable part.
Show that:

(a) for any component E in the support of F, $E^2 < 0$;

(b) $F^2 < 0$.

(vii) Show that $|D|$ is base point free and that the Kodaira dimension of D is one.

3 In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) Let X be an irreducible projective variety of dimension n. Let D be a Cartier divisor on X.
Show that there exists a positive real number C such that

$$h^0(X, \mathcal{O}_X(mD)) \leqslant Cm^n,$$

for $m \gg 0$.

(ii) Let X be an irreducible projective variety of dimension n. Let D be a Cartier divisor on X such that $h^0(X, \mathcal{O}_X(mD)) \geqslant C'm^n$ for some positive real number C' and for infinitely many positive integers m. Let F be an effective divisor.
Show that there exist infinitely many positive integers k such that

$$h^0(X, \mathcal{O}_X(kD - F)) \neq 0.$$

For the purpose of this question you can assume that the statement that you proved in (i) holds for any projective scheme of dimension n.

(iii) Let X be an irreducible projective variety of dimension n. Let D be a Cartier divisor on X
Show that the following are equivalent:

(1) there exists a positive real number C' such that $h^0(X, \mathcal{O}_X(mD)) \geqslant C'm^n$ for and for infinitely many positive integers m;

(2) for some ample Cartier divisor A on X, there exists a positive integer n and an effective divisor E such that $jD \sim A + E$;

(3) for some ample Cartier divisor A on X, there exists a positive integer n and an effective divisor E such that $jD \equiv A + E$;

Show moreover that conditions (2) and (3) can be replaced by the stronger conditions:

(2') for any ample Cartier divisor A on X, there exists a positive integer n and an effective divisor E such that $jD \sim A + E$;

(3') for any ample Cartier divisor A on X, there exists a positive integer n and an effective divisor E such that $jD \equiv A + E$.

(iv) Let X be an irreducible projective variety of dimension n. Let D be a Cartier divisor on X satisfying any of the hypothesis stated in (iii).
Show that there exists a positive integer n such that the natural map induced by the linear system $|jD|$,

$$\phi_{|jD|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(jD))),$$

is birational onto its image. When X is smooth, is the converse true?

(v) Let X be an irreducible projective variety of dimension n. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.
Show that the following are equivalent:

(a) $D = \sum d_i D_i$, where the d_i are positive real numbers and D_i are Cartier divisors on X satisfying any of the properties from part (iii).

(b) for some ample Cartier divisor A on X, there exists a positive real number t and an effective $\mathbb{R}$-divisor E such that $D \sim_{\mathbb{R}} tA + E$;

(c) for some ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A' on X, there exists an effective $\mathbb{R}$-divisor E' such that $D \equiv A' + E'$;

Show moreover that if D' is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X with $D \equiv D'$ then D satisfies any of the properties $(a) - (c)$ if and only if D' does.

(vi) Let X be an irreducible projective variety of dimension n. Show that the set

$$\mathrm{Big}(X) := \{[D] \in \mathrm{N}^1_{\mathbb{R}}(X) \mid D \text{ satisfies any of the properties stated in (v)}\}$$

is an open cone in $\mathrm{N}^1_{\mathbb{R}}(X)$.
Can $\mathrm{Big}(X)$ contain a positive dimensional vector subspace of $\mathrm{N}^1_{\mathbb{R}}(X)$?

END OF PAPER