UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Thursday, 31 May, 2018   9:00 am to 12:00 pm

PAPER 138

MODULAR REPRESENTATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Suppose that k is a field of characteristic p and that p divides the order of the finite group G. Show that kG is not semisimple.

(b) Let k be a field of characteristic p. Recall that a module over a ring is *cyclic* if it is generated by one element. Show that every finitely-generated $k[X]/(X^{p^n})$-module is a direct sum of cyclic modules $M_r = k[X]/(X^r)$, where $1 \leqslant r \leqslant p^n$. Show that each module M_r is uniserial, (i.e. it has a unique composition series) and hence is indecomposable. Identify those M_r which are irreducible.

Deduce that if G is cyclic group of order p^n then kG has exactly p^n indecomposable modules, one of each dimension i with $1 \leqslant i \leqslant p^n$, each being uniserial.

2

(a) Let G be a finite group with $|G| = p^a m$, where p does not divide m, and let k be a field of characteristic p. Suppose that k has all mth roots of unity. Define the *Brauer character* χ_V of the representation $\rho : G \to \mathrm{GL}(V)$. If V, V' are finite-dimensional kG-modules, show that $\chi_V = \chi_{V'}$ if and only if the multiplicities of each simple module as composition factors of V, V' are equal. [Relevant results may be quoted, if stated clearly.]

(b) Let (K, R, k) be a p-modular system and let G be a finite group and let V, V' be finite-dimensional kG-modules, with Brauer characters χ_V and $\chi_{V'}$, respectively. Show that

(i) $\chi_V(g^{-1}) = \overline{\chi_V(g)} = \chi_{V^*}(g)$;

(ii) $\chi_{V \otimes V'} = \chi_V \cdot \chi_{V'}$;

(iii) If χ is a Brauer character, then the trivial Brauer character is a constituent of $\chi\overline{\chi}$.

(c) Suppose that G is a non-abelian simple group and suppose that p is an odd prime. If χ is a non-trivial irreducible p-modular Brauer character of G show that $\chi(1) > 2$.

3

Let G be a finite group and k a field of characteristic $p > 0$.

(a) Define the *Grothiendieck group* $R(G)$ of G and explain how one makes $R(G)$ into a commutative ring.

(b) Show that the map

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \prod \mathbb{C}$$

(over conjugacy classes of p'-elements of G) is an algebra isomorphism. [Relevant results may be assumed, if clearly stated.]

Deduce that the number of simple kG-modules up to isomorphism equals the number of conjugacy classes of p'-elements of G.

(c) Let k be a splitting field of characteristic p for finite groups G_1 and G_2. Deduce from (b) that the simple $k(G_1 \times G_2)$-modules are precisely the tensor products $S_1 \otimes S_2$, where S_i is a simple kG_i-module, $(i = 1, 2)$, and the action of $G_1 \times G_2$ is given by $(g_1, g_2) \cdot (s_1 \otimes s_2) = g_1 s_1 \otimes g_2 s_2$. Moreover show that two such tensor products $S_1 \otimes S_2$ and $S_1' \otimes S_2'$ are isomorphic as $k(G_1 \times G_2)$-modules if and only if $S_i \cong S_i'$ as kG_i-modules, $(i = 1, 2)$.

4

Let $(K, \mathfrak{O}, k)$ be a splitting p-modular system for the finite group G. Assume K and $\mathfrak{O}$ are complete with respect to the valuation. Let V be an irreducible KG-module. Define an *$\mathfrak{O}$-form*, W, for V and use it to define the *decomposition matrix*, D. Define also the *Cartan matrix*, C, of kG.

(i) If M, M' are KG-modules with $\mathfrak{O}$-forms W, W' respectively, show that $\mathrm{Hom}_{\mathfrak{O}G}(W, W')$ is an $\mathfrak{O}$-form of $\mathrm{Hom}_{KG}(M, M')$.

(ii) With the usual notation, show that $\mathfrak{p}\mathrm{Hom}_{\mathfrak{O}G}(W, W') = \mathrm{Hom}_{\mathfrak{O}G}(W, \mathfrak{p}W')$ as a subset of $\mathrm{Hom}_{\mathfrak{O}G}(W, W')$.

(iii) Suppose now that W is a projective $\mathfrak{O}G$-lattice. Show that

$$\mathrm{Hom}_{\mathfrak{O}G}(W, W')/\mathfrak{p}\mathrm{Hom}_{\mathfrak{O}G}(W, W') \cong \mathrm{Hom}_{\mathfrak{O}G}(W, W'/\mathfrak{p}W') \cong \mathrm{Hom}_{kG}(k \otimes_{\mathfrak{O}} W, k \otimes_{\mathfrak{O}} W').$$

Deduce that

$$\dim_K \mathrm{Hom}_{KG}(M, M') = \dim_k \mathrm{Hom}_{kG}(k \otimes W, k \otimes W').$$

(iv) Deduce that $D^T D = C$.

5

If R is a ring with 1, what does it mean for an R-module M to *belong to* a block B? If R is a finite-dimensional semisimple algebra over a field, describe the blocks of R (proofs are not needed).

(a) Let R be a ring with identity. Let $\mathcal{C}_1, \mathcal{C}_2$ be two sets of irreducible R-modules with the property that $\mathcal{C}_1 \cup \mathcal{C}_2$ contains all isomorphism types of irreducible modules, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, and for all $S \in \mathcal{C}_1, T \in \mathcal{C}_2$ there is no non-split extension $0 \to S \to V \to T \to 0$ or $0 \to T \to V \to S \to 0$. Show that every finite length module M can be written as $M = U_1 \oplus U_2$ where U_1 has all its composition factors in $\mathcal{C}_1$ and U_2 has all its composition factors in $\mathcal{C}_2$. Show also that the submodules U_1 and U_2 are the unique largest submodules of M with all composition factors in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. [You may assume the Jordan-Hölder theorem.]

(b) Define an equivalence relation on the set of irreducible modules of an algebra R as follows: define $S \sim T$ if either $S \cong T$ or there is a list of irreducible R-modules $S = S_1, S_2, \ldots, S_n = T$ such that for each $i = 1, 2, \ldots, n-1$, the modules S_i, S_{i+1} appear in a non-split short exact sequence of R-modules $0 \to U \to V \to W \to 0$ with $\{U, W\} = \{S_i, S_{i+1}\}$. You may assume this is an equivalence relation.

Let R be a finite-dimensional algebra over a field k. Show that the following are equivalent for irreducible R-modules S and T:

(i) S and T lie in the same block;

(ii) There is a list of irreducible R-modules $S = S_1, S_2, \ldots, S_n = T$ such that S_i and S_{i+1} are both composition factors of the same projective indecomposable projective module, for each $i = 1, \ldots, n-1$;

(iii) $S \sim T$.

[Hint: for (i) $\implies$ (iii) use part (a).]

6

(a) Let k be a field of characteristic p that is a splitting field for G and all of its subgroups and let D be a p-subgroup of G. Define the *Brauer morphism*, and show (in the usual notation) that the Brauer morphism induces a one-to-one correspondence between block idempotents in $Z(kG)$ with defect group D and primitive idempotents in $(kC_G(D))_D^{N_G(D)}$. State Brauer's first main theorem and deduce it from the previous result.

(b) Let $G = A_5$ and let k be a splitting field of characteristic 2. Using the fact that kA_4 has only one block (which you need not verify), show that the subgroups C_2 cannot occur as defect groups of blocks of kG.

(c) Let $G = S_3$ and $k = \mathbb{F}_4$. Let $N = \langle(123)\rangle$ and $H = \langle(12)\rangle$. It is given that the block idempotents of kN are

$$e_1 = 1 + (123) + (132),\ e_2 = 1 + \omega(123) + \omega^2(132) \text{ and } e_3 = 1 + \omega^2(123) + \omega(132),$$

where ω is a primitive cube root of unity in $\mathbb{F}_4$ (you are not asked to verify this). Compute the orbits of G on this set of idempotents and hence find the block idempotents of $\mathbb{F}_4G$. For each of the block idempotents of $\mathbb{F}_4G$, compute the effect of the Brauer morphisms Br_1^G and Br_H^G. Hence compute the defect groups of each block.

END OF PAPER