UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2018 1:30 pm to 4:30pm

PAPER 137

MODULAR FORMS AND L-FUNCTIONS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function such that for every $n > 0$, $y^n f(y) \to 0$ as $y \to \infty$. What is the *Mellin transform* $M(f,s)$ of f? Find the Mellin transform of the function $1/(e^{\pi y} - 1)$.

Suppose there is an increasing sequence of real numbers $\sigma_1 < \sigma_2 < \dots$ and nonzero constants $c_j \in \mathbb{C}$ such that for every integer $k \geqslant 0$,

$$f(y) = c_1 y^{\sigma_1} + \cdots + c_k y^{\sigma_k} + y^{\sigma_{k+1}} g_k(y),$$

where g is continuous on $[0,\infty)$. Show that $M(f,s)$ has a meromorphic continuation to $\mathbb{C}$, holomorphic apart from a simple pole at $s = -\sigma_j$ with residue c_j for each $j \geqslant 1$.

Compute the residue of $\Gamma(s)$ at $s = -j$ ($j \geqslant 0$). Hence or otherwise show that $\zeta(1-n) = (-1)^{n-1} B_n/n$ for every $n \geqslant 1$, where the Bernoulli numbers B_n are defined by the generating series

$$\frac{t}{e^t - 1} = \sum_{n \geqslant 0} \frac{B_n}{n!} t^n.$$

2 (i) Let G be a finite abelian group, and $\widehat{G}$ the group of characters of G. Show that if $g,\ h \in G$ then

$$\sum_{\chi\in\widehat{G}} \chi(g)^{-1}\chi(h) = \begin{cases} 0 & \text{if } h \neq g \\ \#G & \text{if } h = g. \end{cases}$$

(ii) Define the Dirichlet L-function $L(\chi, s)$ for a character $\chi\colon G = (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, where $N > 1$. Show that if $(a, N) = 1$ then

$$\sum_{\chi\in\widehat{G}} \chi(a)^{-1} L(\chi, s) = \phi(N) \sum_{\substack{n\geqslant 1 \\ n\equiv a \pmod N}} n^{-s}.$$

(iii) Let $k \geqslant 3$, and $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with $\chi(-1) = (-1)^k$. Let

$$G_k(\chi, z) = \sum_{\substack{m,\ n\in\mathbb{Z} \\ (n,N)=1}} \frac{\chi(n)}{(mz+n)^k}.$$

Show that $G_k(\chi, z+N) = G_k(\chi, z)$, and that

$$G_k(\chi, z) = \sum_{n\geqslant 0} c_n e^{2\pi i n z/N},$$

where $c_0 = 2L(\chi, k)$ and for $n \geqslant 1$

$$c_n = 2\frac{(-2\pi i)^k}{(k-1)!N^k} g(\chi) \sum_{d|n} \chi(d)^{-1} d^{k-1},$$

and

$$g(\chi) = \sum_{\substack{1\leqslant a\leqslant N \\ (a,N)=1}} \chi(a) e^{2\pi i a/N}.$$

[*You may use without proof the formula*

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$.]

3 (i) Let $\mathcal{D} \subset \mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the subset defined by the conditions

$$-\frac{1}{2} < \mathrm{Re}(z) \leqslant \frac{1}{2}, \ |z| \geqslant 1, \ \text{and} \ |z| = 1 \implies \mathrm{Re}(z) \geqslant 0.$$

Show that any element of $\mathcal{H}$ is equivalent under $\Gamma = SL(2, \mathbb{Z})$ to a unique element of $\mathcal{D}$, and determine the stabilisers under Γ of the elements of $\mathcal{D}$.

(ii) Explain the meaning of the terms *weak modular form*, *modular form* and *cusp form* of weight k and level 1.

Let f be a weak modular form of weight k and level 1. Show by induction that for every $\ell \geqslant 0$, $f^{(\ell)}(z) = (d/dz)^\ell f(z)$ satisfies

$$f^{(\ell)}(-1/z) = z^{k+2\ell} f^{(\ell)}(z) + \sum_{j=0}^{\ell-1} c_{\ell,j} z^{k+j+l} f^{(j)}(z)$$

where for $0 \leqslant j \leqslant \ell - 1$,

$$c_{\ell,j} = \binom{\ell}{j} (k + \ell - 1)(k + \ell - 2) \cdots (k + j).$$

Deduce that if $f = \sum a_n q^n$ is a weak modular form of weight $k < 0$, then $\sum n^{1-k} a_n q^n$ is a weak modular form of weight $2 - k$.

4 Let $\Gamma = SL(2, \mathbb{Z})$ and let n be a positive integer. Show that the orbits of Γ acting on the set of 2×2 integer matrices with determinant n are parametrised by the set

$$\Pi_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, \ b, \ d \in \mathbb{Z}, \ 0 \leqslant b < d, \ ad = n \right\}.$$

Let $f \in M_k(\Gamma)$. Show that $T_n f = n^{k/2-1} \sum_{\gamma \in \Pi_n} f|_k \gamma$ belongs to $M_k(\Gamma)$, and compute its q-expansion. Deduce that $a_1(T_n f) = a_n(f)$.

Let $\mathbb{T} \subset \mathrm{End}(S_k(\Gamma))$ be the subring generated by $\mathbb{Z}$ and $\{T_n \mid n \geqslant 1\}$. Let $S_k(\Gamma, \mathbb{Z}) = \{f \in S_k(\Gamma) \mid a_n(f) \in \mathbb{Z} \text{ for all } n \geqslant 1\}$. Show that if $T \in \mathbb{T}$ and $f \in S_k(\Gamma, \mathbb{Z})$ then $Tf \in S_k(\Gamma, \mathbb{Z})$.

By using a suitable $\mathbb{Z}$-basis for $S_k(\Gamma, \mathbb{Z})$, show that the map

$$\alpha \colon \mathbb{T} \to \mathrm{Hom}_{\mathbb{Z}}(S_k(\Gamma, \mathbb{Z}), \mathbb{Z})$$

given by $\alpha(T)(f) = a_1(Tf)$ for $T \in \mathbb{T}$, $f \in S_k(\Gamma, \mathbb{Z})$, is an isomorphism of $\mathbb{Z}$-modules, and that a $\mathbb{Z}$-basis for $\mathbb{T}$ is $\{T_1, \ldots, T_m\}$ where $m = \dim S_k(\Gamma)$.

[*You may assume without proof that the forms $\Delta^j E_4^a E_6^b$, with $1 \leqslant j \leqslant m$, $a \geqslant 0$, $b \in \{0, 1\}$ and $12j + 4a + 6b = k$, form a $\mathbb{Z}$-basis for $S_k(\Gamma, \mathbb{Z})$.*]

5 Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index d. Explain carefully what is a modular form of weight k on Γ. Show that for all $k \geqslant 0$, $\dim M_k(\Gamma) \leqslant 1 + kd/12$. [*You may assume the formula for the number of zeros of a modular form of level 1.*]

Let $\Gamma = \Gamma_0(N)$. Suppose that N_1, D are positive integers with $N_1 D | N$. Show that if $f \in M_k(\Gamma_0(N_1))$, then $f(Dz) \in M_k(\Gamma_0(N))$. Find a basis for the space $M_4(\Gamma_0(3))$.

END OF PAPER