UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 1:30 pm to 3:30 pm

PAPER 130

RAMSEY THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Hales-Jewett theorem, and deduce van der Waerden's theorem.

Let $m > 1$ be fixed. A collection F of subsets of $[n]$ is called *adequate* if whenever $[m]^n$ is 2-coloured there is a monochromatic line whose active coordinate set belongs to F.

(i) Show that, for any n, the collection of all subsets of $[n]$ that contain 1 is never adequate.

(ii) More generally, show that, for any n, a collection of subsets of $[n]$ that is intersecting (meaning that any two meet) cannot be adequate.

(iii) Show that, for n sufficiently large, the collection of all subsets of $[n]$ of even size is adequate.

2

State and prove Rado's theorem.

[You may assume that, for any m, p, c, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic (m, p, c)-set.]

Let A be a partition regular matrix with at least two columns. Prove that *either* whenever $\mathbb{N}$ is finitely coloured there is a monochromatic vector x with $x_1 = x_2$ such that $Ax = 0$, *or* whenever $\mathbb{N}$ is finitely coloured there is a monochromatic vector x with $x_1 \neq x_2$ such that $Ax = 0$. (Here as usual x_i denotes the i-th coordinate of x.)

Give examples (of suitable partition regular matrices and suitable colourings) to show that it may not be possible to choose x with $x_1 = x_2$ and that it may not be possible to choose x with $x_1 \neq x_2$.

3

Define the topological space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

Prove that there exists an idempotent ultrafilter in $\beta\mathbb{N}$.

[You may assume that the operation $+$ on $\beta\mathbb{N}$ is associative and left-continuous.]

Let $U_1, U_2, \ldots$ and U be distinct ultrafilters such that U_n tends to U.

(i) Show that there exist subsets $A_1, A_2, \ldots$ of $\mathbb{N}$ such that $A_n \in U_m$ if and only if $m = n$.

(ii) Show that there exist disjoint subsets $B_1, B_2, \ldots$ of $\mathbb{N}$ such that $B_n \in U_m$ if and only if $m = n$.

(iii) By considering the union of the sets $B_2, B_4, B_6, \ldots$, obtain a contradiction to the assumption that U_n tends to U.

(iv) Deduce that every convergent sequence in $\beta\mathbb{N}$ is eventually constant.

4

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *completely Ramsey*?

Give an example of a non-Ramsey set, and an example of a set that is Ramsey but not completely Ramsey.

Prove that every $*$-open set is completely Ramsey.

[Any results quoted from the course must be proved.]

(i) Prove that every basic $*$-open set is $*$-closed.

(ii) Is every $*$-open set $*$-closed?

(iii) Give an example of a (non-empty) $*$-open set that is τ-nowhere-dense.

END OF PAPER