UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2018 1:30 pm to 4:30 pm

PAPER 121

TOPICS IN SET THEORY

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $M \subseteq N$ be sets. State the Tarski-Vaught criterion that guarantees that $(M, \in)$ is an elementary substructure of $(N, \in)$. (No proof needed.)

(ii) State the Mostowski Collapsing Theorem for a set M with a relation $E \subseteq M \times M$. (No proof needed.)

(iii) Give definitions of the notions of *inaccessible cardinal* and *worldly cardinal*.

(iv) Let $M \subseteq N$ be sets and let φ be a formula in n free variables. Give definitions of the notions

(a) φ is downwards absolute between $(M, \in)$ and $(N, \in)$,

(b) φ is upwards absolute between $(M, \in)$ and $(N, \in)$, and

(c) φ is absolute between $(M, \in)$ and $(N, \in)$.

(v) Assume that κ is a worldly cardinal. Construct transitive set models $(M, \in)$ and $(N, \in)$ of ZFC such that

(a) $M \subseteq N$,

(b) the set M is countable; and

(c) the set N has cardinality $\aleph_1$ and for all countable ordinals α, $\alpha \in N$, and $(N, \in) \models$"α is a countable ordinal".

Prove that your models have these properties.

(vi) Assume that κ is a worldly cardinal and use the sets M and N constructed in (v). Give concrete examples of formulas φ_1, φ_2, and φ_3 such that

(a) φ_1 is absolute between $(M, \in)$ and $(N, \in)$,

(b) φ_2 is downwards absolute but not upwards absolute between $(M, \in)$ and $(N, \in)$, and

(c) φ_3 is neither downwards nor upwards absolute between $(M, \in)$ and $(N, \in)$.

Prove that your formulas have these properties.

2

(i) Define the constructible hierarchy ($\mathbf{L}_\alpha\,;\,\alpha \in \mathrm{Ord}$). (You may assume that for each set A and each natural number n, the set $\mathrm{Def}(A, n)$ of internally definable subsets of A^n is already defined.)

(ii) Prove that

(a) for all ordinals α, $\mathbf{L}_\alpha$ is transitive,

(b) for all ordinals $\alpha \leqslant \beta$, $\mathbf{L}_\alpha \subseteq \mathbf{L}_\beta$, and

(c) for all ordinals α, $\mathbf{L}_\alpha \cap \mathrm{Ord} = \alpha$.

(iii) Assume $\mathbf{V}$=$\mathbf{L}$ and show that if M is any uncountable transitive set model of ZFC and α is a countable ordinal, then $(M, \in) \models$"α is countable".

(iv) We write NC for the statement "$\aleph_1^{\mathbf{L}}$ is countable". Assume that $\mathsf{ZFC} + \mathsf{NC}$ is consistent and prove that $\mathsf{ZFC} + \mathsf{NC} + \mathsf{GCH}$ is consistent.

3

In this question, M is a countable transitive model of ZFC, $(\mathbb{P}, \leqslant, \mathbf{1}) \in M$ is a partial order, and G is $\mathbb{P}$-generic over M.

(i) Define a name $\tau \in M^{\mathbb{P}}$ such that $\mathrm{val}(\tau, G) = \mathbb{P} \backslash G$ and give a proof of this fact.

(ii) Let φ a formula of the forcing language $\mathcal{L}_{\in}(M^{\mathbb{P}})$ with one free variable x and assume that for any name $\sigma \in M^{\mathbb{P}}$, the following equivalence is true:

$$M[G] \models \varphi[\tfrac{\sigma}{x}] \iff \text{there is } p \in G \text{ such that } (p \Vdash^{*} \varphi[\tfrac{\sigma}{x}])^{M}.$$

Show that the following are equivalent:

(a) $M[G] \models \exists x \varphi$ and
(b) there is a $p \in G$ such that $(p \Vdash^{*} \exists x \varphi)^{M}$.

(iii) Define what a Δ-system is and state the Δ-system lemma. (No proof needed.)

(iv) (a) Define what it means for a partial order $(\mathbb{P}, \leqslant, \mathbf{1})$ to preserve cardinals.

(b) Define what it means for a partial order $(\mathbb{P}, \leqslant, \mathbf{1})$ to have the countable chain condition.

(c) Prove that every partial order with the countable chain condition preserves cardinals.

(v) Let $f : \mathbb{N} \to \mathbb{N}$ a function. We say that f *bounds* M if for every $g \in M$ such that $g : \mathbb{N} \to \mathbb{N}$ there are infinitely many numbers k such that $g(k) < f(k)$.

Inside M, define $\mathbb{P} := \{(n, s, A)\,;\, n \in \mathbb{N},\ s \text{ is a partial function with } \mathrm{dom}(s) = n, \mathrm{ran}(s) \subseteq \mathbb{N}, \text{ and } A \subseteq \mathbb{N} \text{ is infinite}\}$ with $(n, s, A) \leqslant (m, t, B)$ if $n \geqslant m$, $s \supseteq t$, $A \subseteq B$, and $\{s(i)\,;\, m \leqslant i < n\} \subseteq B$.

Assume that G is $\mathbb{P}$-generic over M and prove that there is a function f that bounds M.

END OF PAPER